Discussion 14B

① Orthonormal Matrices & Projections

$$A^{M\times N} = \begin{array}{c c} & N \\ M & \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \end{array}$$

An orthonormal matrix has column vectors $\vec{a}_j$ that satisfy

$\langle \vec{a}_i, \vec{a}_j \rangle = 0$ if $i \neq j$ and $\|\vec{a}_j\|^2 = \langle \vec{a}_j, \vec{a}_j \rangle = 1$

a) Suppose $A^{M\times N}$ has linearly independent columns, and $\vec{y} \in \mathbb{R}^M$ is not in the span of column vectors $col(A)$.

Write out the expression for projecting $\vec{y}$ onto $col(A)$:

$$\text{proj}_{C(A)}(\vec{y}) = \boxed{A \underbrace{(A^TA)^{-1}A^T\vec{y}}_{\hat{\vec{x}}}}$$

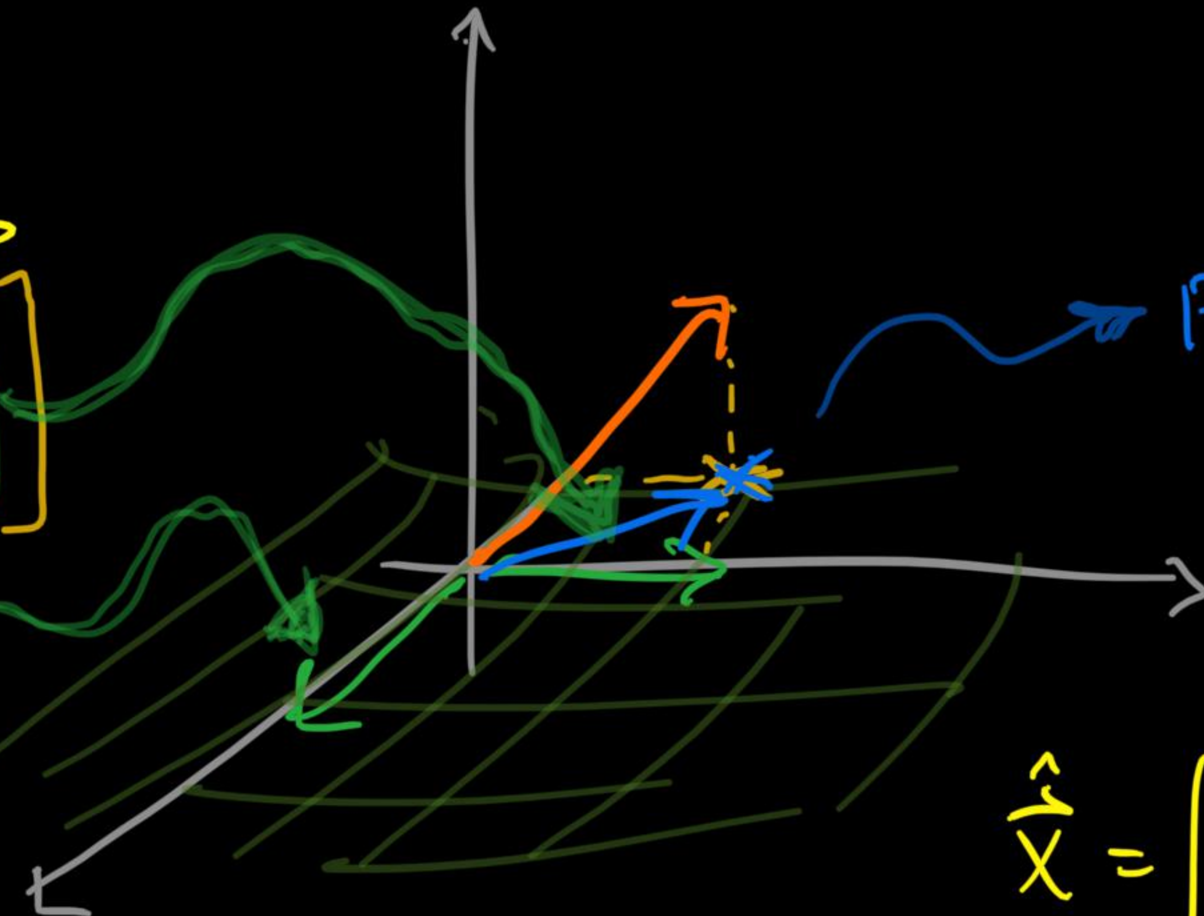
least-squares solution

Projection

Ex:

$$A = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$$\vec{y} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$



$$\text{proj}_{C(A)}(\vec{y}) = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\hat{\vec{x}} = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$A\hat{\vec{x}} = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix} = \text{proj}_{C(A)}(\vec{y})$$

b] Show that a square ($M=N$), orthonormal matrix $A^{N\times N}$ has columns $\vec{a}_j$ which form a basis for $\mathbb{R}^N$.

What is a basis for $\mathbb{R}^N$? It is a set of vectors that are 1. Linearly independent
2. Span the space $\mathbb{R}^N$

Let's show these...

1. Show columns $\vec{a}_j$ are linearly independent, meaning $A\vec{x}=\vec{0}$ implies $\vec{x}=\vec{0}$

Exploit the orthonormal relation...

$A\vec{x}=\vec{0}$

$$\langle \vec{a}_j, A\vec{x}\rangle = \cancelto{0}{\langle \vec{a}_j, \vec{a}_1\rangle} x_1 + \cancelto{0}{\langle \vec{a}_j, \vec{a}_2\rangle} x_2 + \cdots$$

(some general 'j')

$$\cdots + \cancelto{1}{\langle \vec{a}_j, \vec{a}_j\rangle} x_j + \cdots + \cancelto{0}{\langle \vec{a}_j, \vec{a}_N\rangle} x_N$$

$$= x_j \equiv 0 \implies \underline{\underline{x_j = 0}}$$

This implies $x_j = 0$ for all $j \in 1, 2, \ldots, N$.
thus $\vec{x}=\vec{0}$ ✓

$$\langle \vec{a}_j, A\vec{x} \rangle = \cancel{\langle \vec{a}_j, \vec{a}_1 \rangle} x_1 + \cancel{\langle \vec{a}_j, \vec{a}_2 \rangle} x_2 + \cdots$$

(some general 'j')

$$\cdots + \cancelto{1}{\langle \vec{a}_j, \vec{a}_j \rangle} x_j + \cdots + \cancelto{0}{\langle \vec{a}_j, \vec{a}_N \rangle} x_N$$

$$= x_j \equiv 0 \implies \underline{\underline{x_j = 0}}$$

This implies $x_j = 0$ for all $j \in 1, 2, \ldots, N$.
thus $\vec{x} = \vec{0}$ ✓

2. Show that we span $\mathbb{R}^N$, meaning $A\vec{x} = \vec{b}$ has a <u>unique</u> solution $\vec{x}$ for <u>any</u> $\vec{b}$ in $\mathbb{R}^N$.

Since all $\vec{a}_j$ are linearly independent, we know A has an empty null-space that implies A has an inverse (only true for square $A^{N\times N}$).

Thus $\vec{x} = A^{-1}\vec{b}$ exists & is unique. ✓

c] Show for tall $(M \geq N)$, orthonormal matrices $A^{M \times N}$ that $A^T A = \mathbb{I}^{N \times N}$.

Identity

$$A^T A = \begin{bmatrix} \leftarrow \vec{a}_1 \rightarrow \\ \leftarrow \vec{a}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{a}_N \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \cancelto{1}{\langle \vec{a}_1, \vec{a}_1 \rangle} & \cancelto{0}{\langle \vec{a}_1, \vec{a}_2 \rangle} & \cdots & \cancelto{0}{\langle \vec{a}_1, \vec{a}_N \rangle} \\ \cancelto{0}{\langle \vec{a}_2, \vec{a}_1 \rangle} & \cancelto{1}{\langle \vec{a}_2, \vec{a}_2 \rangle} & & \vdots \\ \vdots & & \ddots & \\ & \cdots & & \cancelto{1}{\langle \vec{a}_N, \vec{a}_N \rangle} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = \mathbb{I}^{N \times N}$$

d] From the part above, show for tall ($M \geq N$) orthonormal matrices $A^{M \times N}$ that the projection of $\vec{y}$ onto col(A) becomes $AA^T\vec{y}$:

$$\text{proj}_{C(A)}(\vec{y}) = A(\underbrace{A^TA}_{I})^{-1}A^T\vec{y}$$

$$= AA^T$$

e] Given $A^{4\times 3} = \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ \end{array} \begin{bmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find the least squares $\hat{\vec{x}}$ of $\vec{y} = \begin{bmatrix} 5 \\ 12 \\ 7 \\ 8 \end{bmatrix}$ onto the column space col(A):

$\vec{e} = \| A\hat{\vec{x}} - \vec{y} \|$

Notice...

$$\langle \vec{a}_3, \vec{a}_3 \rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + 0 \cdot 0$$

$$= \frac{1}{2} + \frac{1}{2} = \underline{1} \checkmark$$

$\vec{a}_1, \vec{a}_2, \vec{a}_3$ are orthonormal!

$$\hat{\vec{x}} = A^T \vec{y}$$

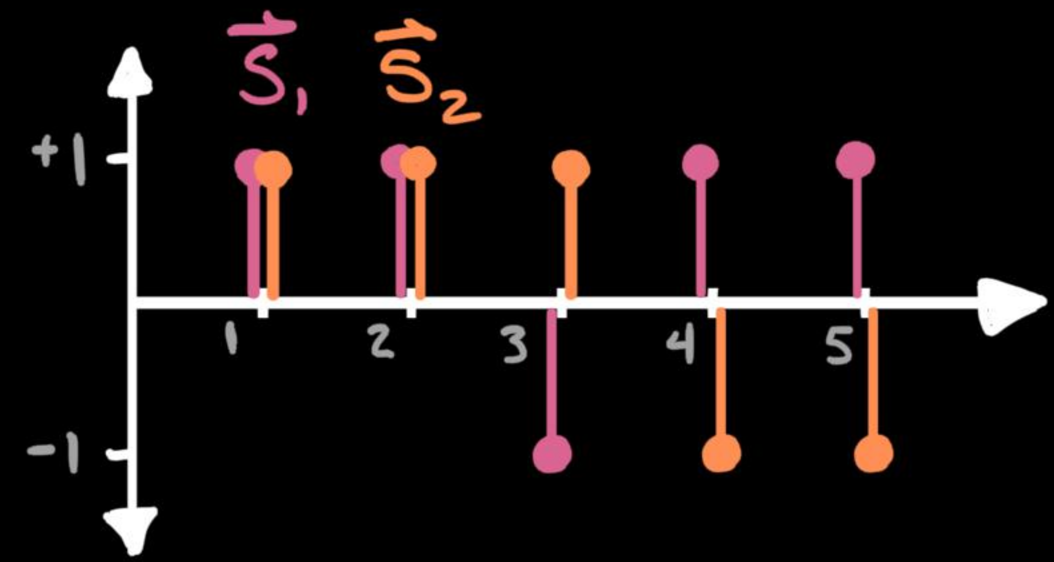
$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ \underline{5/\sqrt{2} + 12/\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 7 \\ 17/\sqrt{2} \end{bmatrix}$$

② Satelite Delays

Suppose we have 2 satellites which communicate by sending a signal $\vec{s}_1$ and $\vec{s}_2$ (respectively) to your phone.

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{s}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$



Your phone receives some signal $\vec{r}$, which came from one of the satellites with a bit of noise. You can estimate which satellite sent the message and the delay by identifying the peak value of

$\text{Corr}_{\vec{r}}(\vec{s}_1)$ and $\text{Corr}_{\vec{r}}(\vec{s}_2)$!

a) Provided $\vec{r}$ (to the right,) determine which satellite it came from and at what delay 'k':

$$\vec{r} = \begin{bmatrix} 1/5 \\ 1/5 \\ 1 \\ 1 \\ -6/5 \\ 1 \\ 1 \\ 1/5 \\ -1/5 \end{bmatrix} \in \mathbb{R}^9$$

Recall: $\text{Corr}_{\vec{a}}(\vec{b})[k] = \sum_n \vec{a}[n]\,\vec{b}[n-k]$

$$\vec{s}_1 = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \qquad \vec{s}_2 = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ \vdots \end{bmatrix}$$

Compute...

$\vdots$ $\text{Corr}_{\vec{r}}(\vec{s}_2)[k]$:

For each of the Correlations, there are non-zero inner-products for $k = -4, -3, -2, \ldots, 7, 8$. That's computing $2 \times 13 = 26$ inner-products for this question alone!!!

$$\text{Corr}_{\vec{r}}(\vec{s}_1)[-4] = \begin{bmatrix} \vdots \\ 0 \\ 1/5 \\ 1/5 \\ 1 \\ 1 \\ -6/5 \\ 1 \\ 1 \\ 1/5 \\ -1/5 \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad 1/5 \quad = 1/5$$

$$\text{Corr}_{\vec{r}}(\vec{s}_2)[-4] = \begin{bmatrix} \vdots \\ 0 \\ 1/5 \\ 1/5 \\ 1 \\ 1 \\ -6/5 \\ 1 \\ 1 \\ 1/5 \\ -1/5 \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ \vdots \end{bmatrix} \quad -1/5 \quad = -1/5$$

$$\mathrm{Corr}_{\vec{f}}(\vec{s}_1)[-3] = \begin{bmatrix} \vdots \\ 0 \\ 1/5 \\ 1/5 \\ 1 \\ 1 \\ -6/5 \\ 1 \\ 1 \\ 1/5 \\ -1/5 \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \qquad 1/5 + 1/5 = 2/5$$

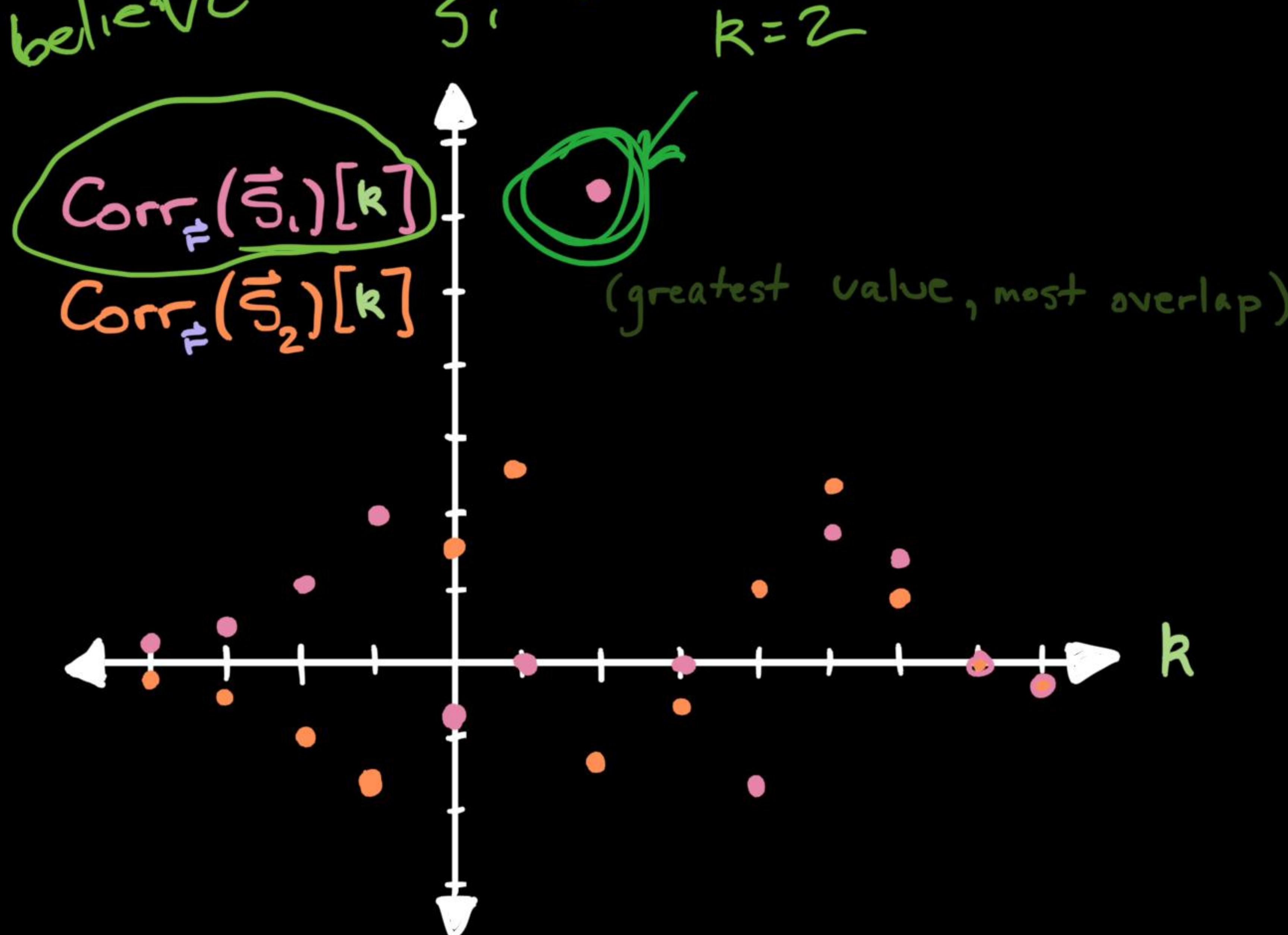
$$\mathrm{Corr}_{\vec{f}}(\vec{s}_2)[3] = \begin{bmatrix} \vdots \\ 0 \\ 1/5 \\ 1/5 \\ 1 \\ 1 \\ -6/5 \\ 1 \\ 1 \\ 1/5 \\ -1/5 \\ 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ \vdots \end{bmatrix} \qquad -1/5 + -1/5 = -2/5$$

(Many years later...)

... we get all the terms!

Then we plot the data:

We believe $\vec{r}$ came from $\vec{s}_1$ with delay $k=2$



b) Provided $\vec{r}$ (to the right), determine which satellite it came from and at what delay 'k':

New $\vec{r}$ received!

$$\vec{r} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 2 \\ 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} \in \mathbb{R}^9$$

Play the exact same game...

Uncertain which satellite sent $\vec{r}$...