EECS 16A

Logistics

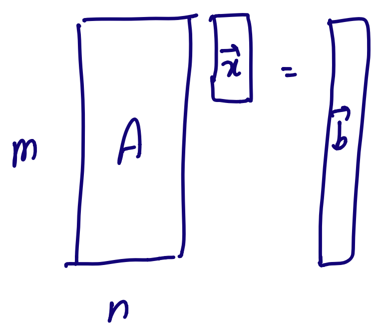
- Happy Thanksgiving!

Today.

- Examples of least squares.
- Trilateration
- Polynomial fitting.
- Linear regression

Least squares:

$A\vec{x} = \vec{b}$



$m > n$

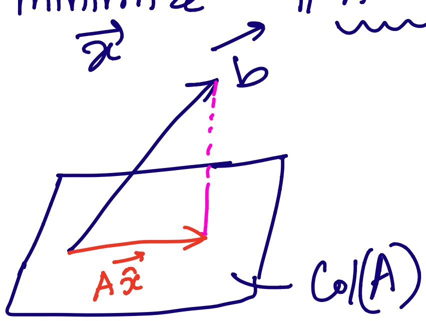
more eqns than unknowns

"Overdetermined system"

Real-world: Noise

$\hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b} \rightarrow$ approximate solution

$\underset{\vec{x}}{\text{minimize}} \; \|A\vec{x} - \vec{b}\|^2 \rightarrow$ minimizing solution $\hat{\vec{x}}$.



$$\textcircled{1}\ \|\vec{x}-\vec{a}\|^2 = d_1^2$$
$$\textcircled{2}\ \|\vec{x}-\vec{b}\|^2 = d_2^2$$
$$\textcircled{3}\ \|\vec{x}-\vec{c}\|^2 = d_3^2$$
$$\textcircled{4}\ \|\vec{x}-\vec{e}\|^2 = d_4^2.$$

$$-2\langle\vec{x},\vec{a}\rangle + 2\langle\vec{x},\vec{c}\rangle = d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2$$

$$-2\langle\vec{x},\vec{a}\rangle + 2\langle\vec{x},\vec{b}\rangle = d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2$$

$$-2\langle\vec{x},\vec{a}\rangle + 2\langle\vec{x},\vec{b}\rangle = d_1^2 - d_4^2 - \|\vec{a}\|^2 + \|\vec{d}\|^2.$$

(See note on Nov 17th)

$$\begin{bmatrix} -2\vec{a}^T + 2\vec{c}^T \\ -2\vec{a}^T + 2\vec{b}^T \\ -2\vec{a}^T + 2\vec{e}^T \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2 \\ d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2 \\ d_1^2 - d_4^2 - \|\vec{a}\|^2 + \|\vec{e}\|^2. \end{bmatrix}$$

Example: $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

→ Gaussian Elimination.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

→ No solutions

Does $\vec{x}$ s.t. $A\vec{x} = \vec{b}$ exist?

Least - squares solution:

$$\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{b}$$

$$(A^T A)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\vec{\hat{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$$

$\vec{\hat{x}}$ is the least squares solution to.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{\hat{x}} = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix} \qquad A\vec{\hat{x}} \neq \vec{b}$$

$\vec{\hat{x}}$ is the soln. that minimize $\|A\vec{x} - \vec{b}\|^2$ ($\|\vec{e}\|^2$)

$\vec{x}$

How did Gauss find Ceres ?

(i) Kepler's laws. — Elliptical orbit.

$$ax^2 + by^2 + cxy + dx + ey = 1$$

are eq^n of ellipses.

$a, b, c, d, e,$: Coefficients : Unknown

x, y : positions

$$\underbrace{(x_1, y_1) \quad (x_2, y_2) \quad \cdots \quad (x_n, y_n)}_{\text{Known}}$$

$(1, 2), \quad (8, 12) \quad \cdots \quad (123, 86)$

Unknowns $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \vec{w}$

$$a x_1^2 + b y_1^2 + c x_1 y_1 + d x_1 + e y_1 = 1$$

$$\begin{bmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 \\ \vdots & \vdots & \vdots & & \\ x_{22}^2 & y_{22}^2 & x_{22}y_{22} & x_{22} & y_{22} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Piazzi

"features"

22 equations
5 unknowns.

$$A\vec{w} = \vec{b}$$

$$\hat{\vec{w}} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

Plug these in to

$$ax^2 + by^2 + cxy + dx + ey = 1$$

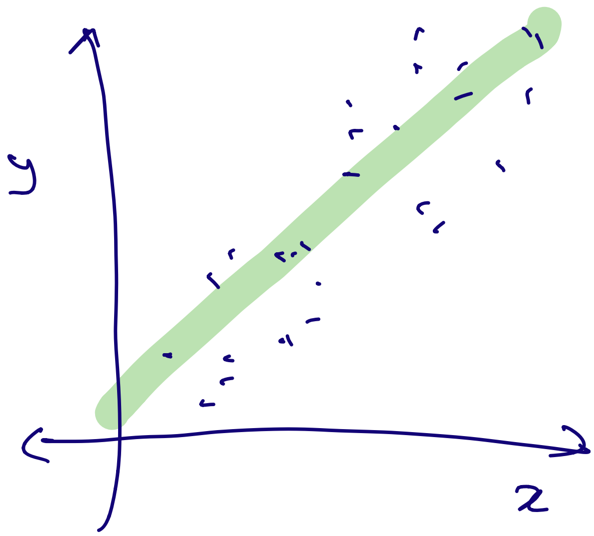
$\boxed{\text{Plot}} \rightarrow$ PREDICTION!!!

Linear regression

$y = mx + c$

m: slope

c: intercept



Known: (x_1, y_1)

(x_2, y_2)

$\vdots$

(x_n, y_n)

$y_1 = m x_1 + c$

$y_2 = m x_2 + c$

$\vdots$

Unknown: m, c $\begin{bmatrix} m \\ c \end{bmatrix}$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{\vec{w}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{b}}$$

"Estimate" of $\vec{\hat{\omega}}$ going to be?

$$\vec{\hat{\omega}} = (A^T A)^{-1} A^T \vec{b} \quad \text{Least squares.}$$

---

(1) Polynomial fitting is powerful

(2) Least-squares is also powerful

$$\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{b}$$

$A^T A \longrightarrow$ might not be invertible!!!