EECS 16A Module 3 - Lecture 6.

- Group formation
- Survey

Today

- Least squares + trilateration
- When does least squares fail?
- $\text{Null}(A^TA) = \text{Null}(A)$
- Model fitting, test data, validation data, training data.

LINEAR in $\vec{x}$

① $\|\vec{x}-\vec{a}\|^2 = d_1^2$

② $\|\vec{x}-\vec{b}\|^2 = d_2^2$

③ $\|\vec{x}-\vec{c}\|^2 = d_3^2$

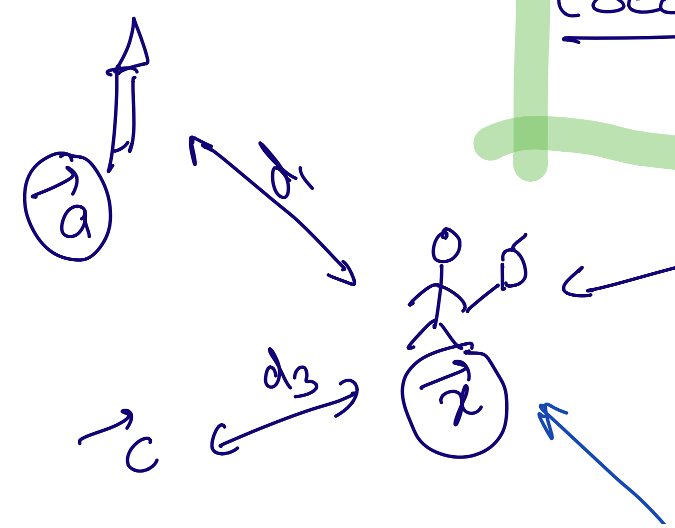
④ $\|\vec{x}-\vec{e}\|^2 = d_4^2$.

$$-2\langle\vec{x},\vec{a}\rangle + 2\langle\vec{x},\vec{c}\rangle = d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2$$

$$-2\langle\vec{x},\vec{a}\rangle + 2\langle\vec{x},\vec{b}\rangle = d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2$$

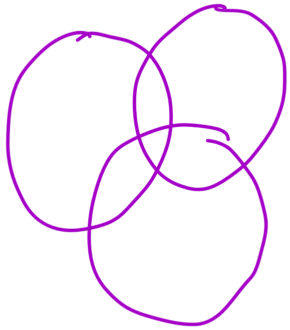
$$-2\langle\vec{x},\vec{a}\rangle + 2\langle\vec{x},\vec{e}\rangle = d_1^2 - d_4^2 - \|\vec{a}\|^2 + \|\vec{e}\|^2.$$

(See note on Nov 17th)



Trilateration algorithm

$d_4 \searrow \vec{e}$



$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \vec{y}^T \vec{x}$$

$$\rightarrow -2\vec{a}^T \vec{x} + 2\vec{c}^T \vec{x} = d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2$$

$$\Rightarrow \left(-2\vec{a}^T + 2\vec{c}^T\right)\left(\vec{x}\right) = \text{—} \,''\, \text{—}$$

$$\underbrace{\begin{bmatrix} -2\vec{a}^T + 2\vec{c}^T \\ -2\vec{a}^T + 2\vec{b}^T \\ -2\vec{a}^T + 2\vec{e}^T \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vec{x} \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2 \\ d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2 \\ d_1^2 - d_4^2 - \|\vec{a}\|^2 + \|\vec{e}\|^2 \end{bmatrix}}_{\vec{b}}$$

$$\Rightarrow \hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}$$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-2\vec{a}^T + 2\vec{c}^T =$$

$$= -2\,[a_1 \quad a_2] + 2\,[c_1 \quad c_2]$$

$$= [-2a_1 + 2c_1 \qquad -2a_2 + 2c_2]$$

$$\begin{bmatrix} -2a_1 + 2c_1 & -2a_2 + 2c_2 \\ -2a_1 + 2b_1 & -2a_2 + 2b_2 \\ -2a_1 + 2e_1 & -2a_2 + 2e_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

How do we know if $(A^TA)^{-1}$ exists?

What if it does not???

Invertibility $\longleftrightarrow$ Nullspaces

Square Matrix Q is invertible iff and only if $\text{Null}(Q) = \{\vec{0}\}$. (Nullspace is trivial).

Idea: Explore $\text{Null}(A^T A)$ to understand invertibility of $A^T A$.

- Thm: $\text{Null}(A^T A) = \text{Null}(A)$

Some reminders before the proof:

① Property of transposes.

$$(AB)^T = B^T A^T$$

$A : n \times m$

$B : m \times k$

$AB : n \times k$

$(AB)^T : k \times n$

$A^T : m \times n$

$B^T : k \times m$

Is $A^T B^T$ a valid multiplication?

Is $B^T A^T$ a valid multiplication?

$B^T A^T : k \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

② Thm: If $\|\vec{x}\| = 0$, then $\vec{x} = \vec{0}$

Proof: $\|\vec{x}\| = 0$

$\|\vec{x}\|^2 = 0$

$x_1^2 + x_2^2 + \cdots + x_n^2 = \langle \vec{x}, \vec{x} \rangle = 0$

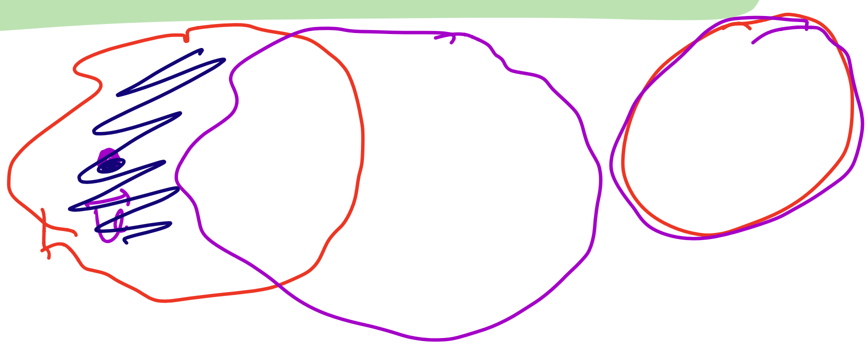
$$\underbrace{x_1^2}_{\geq 0} + \underbrace{x_2^2}_{\geq 0} + \cdots + \underbrace{x_n^2}_{\geq 0} = 0$$

$x_1^2 = 5 \longrightarrow$ not possible

The only way to satisfy the equation is to have all $x_1 = x_2 = \cdots x_n = 0$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \vec{x} \text{ is the } 0 \text{ vector.}$$

$$\boxed{\text{Thm: } \quad \text{Null}(A^TA) = \text{Null}(A)}$$



→ Say $\vec{v} \in \text{Null}(A^TA)$, then it had also better belong to $\text{Null}(A)$

Say $\vec{w} \in Null(A)$, then it had also better belong to $Null(A^TA)$

Formally:

(i) If $\vec{w} \in Null(A)$, then $\vec{w} \in Null(A^TA)$

(ii) If $\vec{v} \in Null(A^TA)$, then $\vec{v} \in Null(A)$.

(i) Known: $\vec{w} \in Null(A)$

$$A \cdot \vec{w} = \vec{0} \quad (1)$$

Want: $\vec{w} \in Null(A^TA)$

$$(A^TA)\vec{w} = \vec{0}$$

Multiply by A^T.

$$A^T(A\vec{w}) = A^T \cdot \vec{0} = \vec{0}$$

$$A^TA \cdot \vec{w} = \vec{0}$$

$$(A^TA)\vec{w} = \vec{0} \Rightarrow \vec{w} \in Null(A^TA)$$

(ii) Known: $\vec{v} \in N(A^TA)$

$A^TA\vec{v} = \vec{0}$

Want: $\vec{v} \in Null(A)$

$A\vec{v} = \vec{0}$

Consider: $||A\vec{v}||^2$

$= \langle A\vec{v}, A\vec{v} \rangle$

$= (A\vec{v})^T(A\vec{v})$

$= \vec{v}^T A^T (A\vec{v})$

$= \vec{v}^T(A^TA\vec{v})$

$= \vec{v}^T \vec{0}$

$= 0$

$\Rightarrow ||A\vec{v}||^2 = 0 \Rightarrow ||A\vec{v}|| = 0$

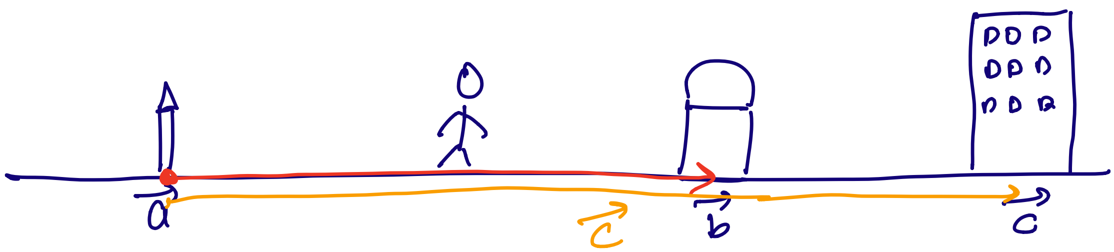
$\Rightarrow A\vec{v} = \vec{0}$

A : Columns of A are linearly dependant.

$\longrightarrow$ Null (A) $\Rightarrow$ non-trivial.

Col's of A are lin. independant

Null (A) $\Rightarrow$ trivial.



$\vec{a} = \vec{0}$

$\vec{b} = \vec{b}$

$\vec{c} = 2 \cdot \vec{b}$

$$2\begin{bmatrix} \vec{b}^T - \vec{a}^T \\ \vec{c}^T - \vec{a}^T \end{bmatrix} \vec{x} = \begin{bmatrix} d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2 \\ d_1^2 - d_3^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2 \end{bmatrix}$$

$$\Rightarrow 2\begin{bmatrix} \vec{b}^T \\ 2\vec{b}^T \end{bmatrix} \vec{x} = \begin{bmatrix} d_1^2 - d_2^2 + \|\vec{b}\|^2 \\ d_1^2 - d_3^2 + 4\|\vec{b}\|^2 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

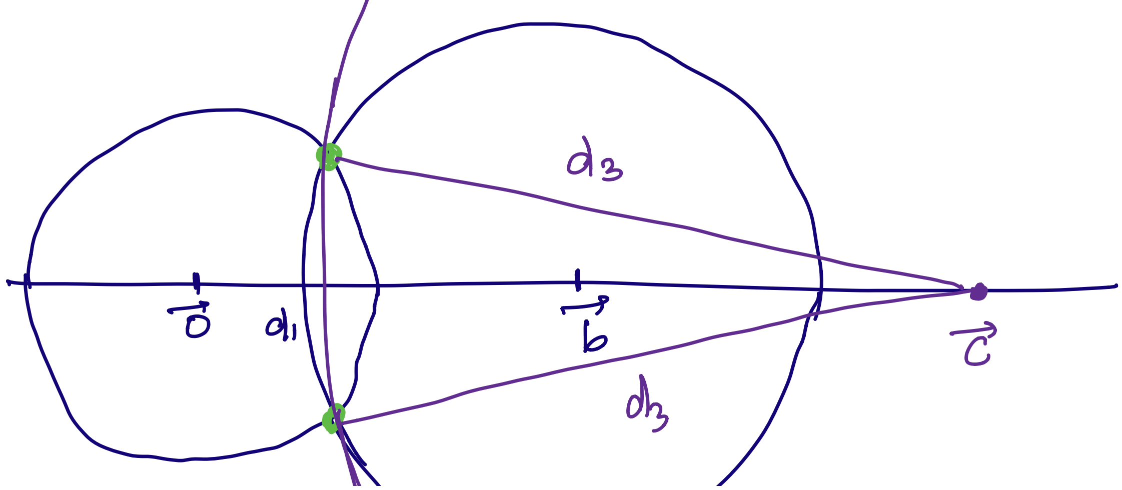
$$\Rightarrow \underbrace{2\begin{bmatrix} b_1 & b_2 \\ 2b_1 & 2b_2 \end{bmatrix}}_{A} \vec{x} = \underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{\vec{b}}$$

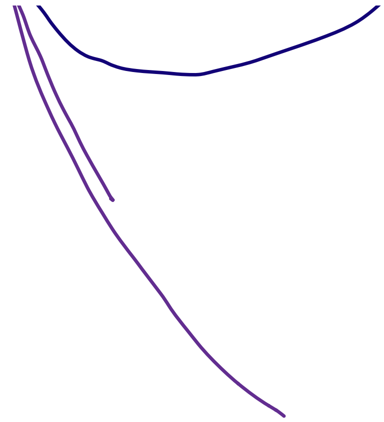
$\Rightarrow$ A lin dep. columns

$\Rightarrow$ A has non-trivial nullspace

$\Rightarrow$ (A^TA) — " — " —

(A^TA) is NOT invertible!





Machine learning : Using data to understand the ~~[illegible]~~ real world.

↳ Making a model.

→ How do you know that your model is any good ?!!