EECS 16A

Module 1 Lecture 3

Today

1. Gaussian Elimination.
2. Matrix- vector multiplication
3. Span.

Logistics

1. OH - See Piazza post. Configured for you to meet each other
2. Try different discussions
3. Python bootcamp is hard for everyone.

Question: Know in advance - are my measurements good? Will they give me a unique solution?

Example:

$$2x_2 + 2x_3 = 2.$$
$$x_1 + x_2 + x_4 = 3$$

4 unknowns.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x}$$

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Matrix Vector

Matrix

Augmented matrix form:

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 3 \end{array}\right]$$

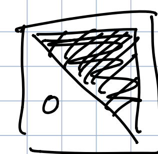
Augmented

$$\xrightarrow{\text{Swap}(R_1,R_2)} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 2 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Row echelon form

Notice: Row 1 is already normalized.
Row 2 does not have x_1,
x_1 is already eliminated.

$\boxed{\text{JARGON}}$ Row echelon form. ←



- All non-zero rows above all zero rows.
- Leading co-ef of a non-zero row is to the right of the row above it.
- Leading coef to be 1.

Reduced row echelon form:

- Each column with a leading 1 has 0's everywhere else

$$\xrightarrow{R_1 - R_2} \quad \text{RREF} \rightarrow \underbrace{\begin{array}{cc} 1 & 2 \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array}}_{\text{Row reduced echelon form.}}$$

$$\begin{bmatrix} 0 & 8 & 0 & 0 \\ 0 & 0 & 9 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Why do we care?

$$\begin{array}{cccc|c} \multicolumn{5}{c}{\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]} \\ x_1 & x_2 & x_3 & x_4 & \end{array}$$

Leading coef: first non-zero elem.
↳ Pivot

x_1, x_2: have a leading 1 in their column
$\Longrightarrow$ Basic variables

$x_3, x_4 \Longrightarrow$ Have no leading 1 in the column
$\Longrightarrow$ Free variables.

$$2x_2 + 2x_3 = 2$$

$$x_1 + x_2 \qquad + x_4 = 3$$

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Steps}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Last row: $x_2 + x_3 = 1$

$x_3 = t$ free variable $x_3 =$ anything.

$x_2 = 1 - x_3 = 1 - t$

$x_1 - \overset{t}{x_3} + x_4 = 2$

x_4 is also a free variable.

Let $x_4 = p$.

$x_1 - t + p = 2$

$x_1 = 2 - p + t$

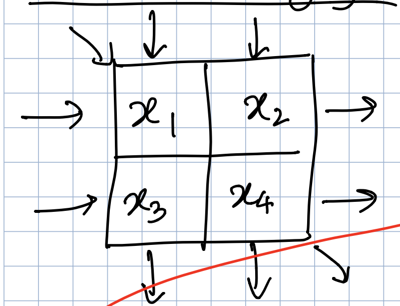
Once I fix x_3, x_4, I have fixed x_1, x_2.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-p+t \\ 1-t \\ t \\ p \end{bmatrix}$$

"Parametric solution"

$$= \begin{bmatrix} 2 - x_4 + x_3 \\ 1 - x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

Back to imaging



$$
\begin{aligned}
x_1 + x_2 \qquad\qquad\qquad &= b_1 \quad (1)\\
x_3 + x_4 &= b_2 \quad (2)\\
x_1 + \qquad x_3 \qquad\quad &= b_3 \quad (3)\\
x_2 \qquad + x_4 &= b_4 \quad (4)\\
x_1 + \qquad\qquad x_4 &= b_5 \quad (5)
\end{aligned}
$$

Eq (1) - (4) were not enough

Measurements (1) - (4)

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & b_1\\
0 & 0 & 1 & 1 & b_2\\
1 & 0 & 1 & 0 & b_3\\
0 & 1 & 0 & 1 & b_4
\end{array}\right]
$$

Do Gaussian Elimination.

$$
\to \left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & b_1\\
0 & 0 & 1 & 1 & b_2\\
0 & -1 & 1 & 0 & b_3-b_1\\
0 & 1 & 0 & 1 & b_4
\end{array}\right]
\to \left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & b_1\\
0 & 1 & 0 & 1 & b_4\\
0 & 0 & 1 & 1 & b_4+b_3-b_1\\
0 & 0 & 1 & 1 & b_2
\end{array}\right]
$$

$$
\to \left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & b_1\\
0 & 1 & 0 & 1 & 4\\
0 & 0 & 1 & 1 & b_4+b_3-b\\
0 & 0 & 0 & 0 & b_2-b_4-b_3+b_1
\end{array}\right]
$$

$[0 \; 0 \; 0 \cdots \mid 0]$

$0x + 0y = 8$

$$\underbrace{\qquad\qquad}_{=0}$$

If $b_2 - b_4 - b_3 + b_1 = 0$ then infinite solⁿ. check

if $\neq 0$ then inconsistent system.

①, ②, ③, ⑤.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 1 & b_2 \\ 1 & 0 & 1 & 0 & b_3 \\ 1 & 0 & 0 & 1 & b_5 \end{array}\right] \xrightarrow{GE} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & b_1 - b_3 \\ 0 & 0 & 1 & 1 & b_5 - b_3 \\ 0 & 0 & 0 & 1 & \frac{b_2 + b_5 - b_3}{2} \end{array}\right]$$

- We can use GE to understand if a system has a unique solution.

- Now, we want to find other (better) to understand if system has unique solⁿ

Matrix - vector multiplication

$\mathbb{R}$ = set of real numbers

$$\begin{array}{c} \\ 2 \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\xleftrightarrow{3}$$

$A \in \mathbb{R}^{2\times 3}$ $\quad \vec{x} \in \mathbb{R}^{3\times 1}$ $\quad \vec{b} \in \mathbb{R}^{2\times 1}$

$$\underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}}_{\text{vector}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\text{vector}}$$

Each row represents a measurement

"Row" perspective or "measurement perspective".

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}$$

Reminder: ① $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$ Addition

② $c \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$ $\quad c \in \mathbb{R}$ real number.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \\ \\ \end{bmatrix} = \vec{a_1}x_1 + \vec{a_2} \cdot x_2 + \vec{a_3} \cdot x_3$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{31}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ a_{32}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \\ a_{33}x_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$