EECS 16A

Module 1, Lecture 4

Logistics
1. HW2 out.
2. Study groups out.
3. Stay Healthy.

Today:
1. Span
2. Linear dependance
3. Linear independance.

} Another way to understand systems of linear equations.

4. Introduction to Proofs

Recap:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\vec{b}} \qquad A\vec{x} = \vec{b}$$

Rows represent "experiments"

a_{ij} represent weight on variable j (x_j) in experiment i

Columns represent the weight on an variable (unknown)
↳ Column j corresponds to x_j.

"Column perspective on Matrix-Vector multiplication"

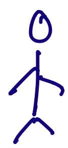
$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \qquad \vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Then $A\vec{x} = \vec{a}_1 \cdot x_1 + \vec{a}_2 \cdot x_2 + \vec{a}_3 \cdot x_3$.

"linear combination of the columns of A"

Recall: Gaussian Elimination steps only depend on the entries of the matrix A and not on the entries of the vector $\vec{b}$.

Imaging: **Question:** How do we know if A is bad?

 $\longrightarrow$ Imaging experiment $\longrightarrow$ Measurement

A $\qquad\qquad$ $\vec{b}$

Object

$\vec{x}$

$\boxed{A\vec{x} = \vec{b}.}$ $\qquad$ $\boxed{\text{Use } \vec{b} \text{ to recover } \vec{x}}$

Today: (1) A new perspective on systems of linear equations.

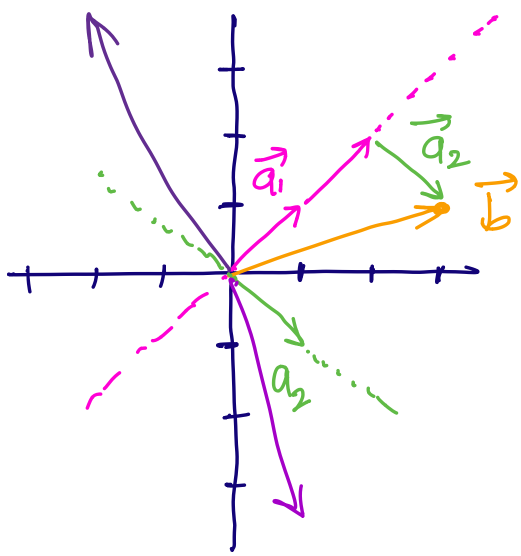
$$A\vec{x} = \vec{b}$$

Previously: Find $x_1, x_2 \ldots x_n$ that satisfy our equations.

Now: What linear combinations of the columns of matrix A can produce the vector $\vec{b}$?

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\qquad$ $\vec{a_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{a_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$A\vec{x} = \vec{b}$ $\qquad$ $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



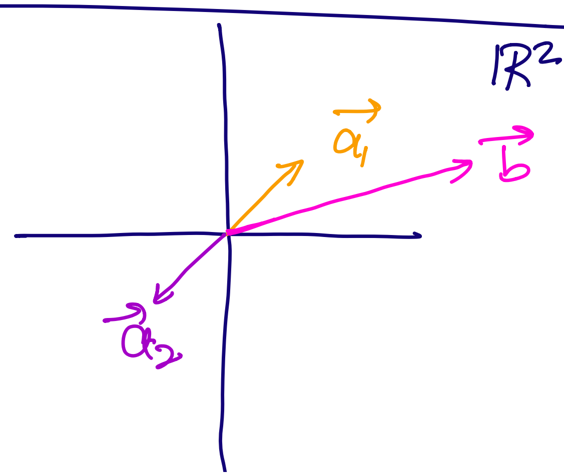
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$x_1 = 2$, $x_2 = 1$

<u>Soln:</u> Take 2 steps along $\vec{a_1}$

1 step along $\vec{a_2}$.

$$A = \begin{array}{c} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ \vec{a_1} \quad \vec{a_2} \end{array}$$



Span: Span of the columns of a Matrix A.
is the set of all vectors $\vec{b}$ such that
$A\vec{x} = \vec{b}$ has a solution. (need not be unique)

→ **i.e.** set of all vectors $\vec{b}$ that can be expressed as linear combinations of the columns of A.

e.g:

$$\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$$

$$= \left\{\vec{v} \;\middle|\; \vec{v} = \alpha\begin{bmatrix}+1\\+1\end{bmatrix}, \alpha \in \mathbb{R}\right\}$$

↑ such that

Range (A) = Columnspace (A)
= Span (Columns (A)).

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$$

Linear combinations of the columns.

$$\vec{v} = \alpha \begin{bmatrix} \vec{a}_1 \end{bmatrix} + \beta \begin{bmatrix} \vec{a}_2 \end{bmatrix}, \alpha, \beta \in \mathbb{R}$$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \underbrace{(\alpha - \beta)}_{\text{real number}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>Consider:</u> $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$= \left\{ \vec{v} \;\middle|\; \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

$$\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$$

entire plane.

$\boxed{\underline{\text{Theorem:}} \quad \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2}$

<u>Proof:</u>

① Beginning | Known

$\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

= set of all $\vec{b}$ that can be written as

$\alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$

$$= \underbrace{\left\{\vec{v} \;\middle|\; \vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix},\ \alpha,\beta\in\mathbb{R}\right\}}$$

$= S$

② "End" / <u>To show:</u>

$\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

If I have any vector $\vec{b} \in \mathbb{R}^2$, $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$, I can reach it using a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$

<u>Want:</u> all $\vec{b}$ to belong to the set S

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

known/fixed

unknown.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}}$$

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -2 & b_2 - b_1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_1 - b_2}{2} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1 + b_2}{2} \\ 0 & 1 & \frac{b_1 - b_2}{2} \end{array}\right]$$

$$\boxed{\alpha = \frac{b_1 + b_2}{2}, \quad \beta = \frac{b_1 - b_2}{2}}$$

$\Rightarrow$ Every $\vec{b} \in \mathbb{R}^2$ can be written as a linear combination!

$\Rightarrow \vec{b} \in S$.   "Constructive Proof"

Definition: **Linear dependance**

A set of vectors $\{\vec{a_1}, \vec{a_2} \cdots \vec{a_n}\}$ are said to be linearly dependant if one of the vectors can be written as a linear combination of the others.

↳ if one of the vectors is in the "span" of the other vectors!

e.g.:

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} \Rightarrow$ are linearly dependant

$\Rightarrow \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eg: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \rightarrow$ are <u>not</u> linearly dependant e.g: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ linearly dependant.

Thm: To reach $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ use

$$\alpha = \frac{b_1 + b_2}{2}, \quad \beta = \frac{b_1 - b_2}{2}$$

$b_1 = 3$, $b_2 = 1$.

$$\alpha = \frac{3+1}{2} = 2 \qquad \beta = \frac{3-1}{2} = 1$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

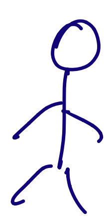
$$= \begin{bmatrix} 2+1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\text{e.g.}\left\{\underbrace{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}}_{\text{span}=\mathbb{R}^2},\underbrace{\begin{bmatrix}31.832157\\-13.5682\end{bmatrix}}_{\in\mathbb{R}^2}\right\} \text{ Linearly dependent.}$$

$$\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$$

Because ~~B~~ we proved theorem.

<u>Linear independance</u>: If vectors $\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\}$ are NOT linearly dependant, then they are linearly independant.



$\longrightarrow A \longrightarrow \vec{b}$

$$A\vec{x} = \vec{b}$$

Thm: Consider $A\vec{x} = \vec{b}$.

p

If the columns of matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does NOT have a unique solution.

q

If $\underbrace{\text{"stuff"}}_{\substack{\text{Known} \\ \text{"beginning"}}}$ then $\underbrace{\text{"stuff"}}_{\substack{\text{to show.} \\ \text{end}}}$

If p then q

__To show:__ $p \underset{\text{implies}}{\Longrightarrow} q$

"16A style" $\longrightarrow$ DO A SIMPLE EXAMPLE.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{array}{cccc} \vec{a_1} & \vec{a_2} & \vec{a_3} & \\ \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Known / Beginning:

$\vec{a_1}, \vec{a_2}, \vec{a_3}$ are L.D.

Some column.

Let us say

$$\vec{a_1} = c_2 \cdot \vec{a_2} + c_3 \vec{a_3}$$

"Without loss of generality"

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{a_1}$$

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \end{bmatrix} \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} = c_2 \cdot \vec{a_2} + c_3 \cdot \vec{a_3}$$

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}$$

Rewrite of pink equation

To show:

$A\vec{x} = \vec{b}$ does not have a unique solution

$\vec{x}_*$ is a solution.

I want to find another solution.

Can do similar proof assuming:

$$\vec{a_2} = c_1 \vec{a_1} + c_3 \vec{a_3}$$

$$\begin{bmatrix} c_1 \\ -1 \\ c_3 \end{bmatrix}$$

$$A\left(\begin{bmatrix}0\\c_2\\c_3\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=0$$

Rearrange

$$\Rightarrow \boxed{A\underbrace{\begin{bmatrix}-1\\c_2\\c_3\end{bmatrix}}=0}$$

is this $\vec{0}$

$\vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Is $\begin{bmatrix}-1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$?

NO

are distinct

Consider:

$$\vec{x}_* + \begin{bmatrix}-1\\c_2\\c_3\end{bmatrix} = \vec{y}$$

$$A\vec{y} = A\left(\vec{x}_* + \begin{bmatrix}-1\\c_2\\c_3\end{bmatrix}\right)$$

$$= A\cdot\vec{x}_* + A\begin{bmatrix}-1\\c_2\\c_3\end{bmatrix}$$

$$= \vec{b} + 0$$

$$= \vec{b}$$

$\Longrightarrow$ $\vec{y}$ is another solution $\Rightarrow \vec{x}_*$ is NOT a unique solution

OFFICE HOURS NOTES

If $\vec{x}_*$ is a solution

$$\vec{a}_1 = c_2\vec{a}_2 + c_3\vec{a}_3$$

$$\Rightarrow \vec{a}_1 - c_3\vec{a}_3 = c_2\vec{a}_2$$

$$\Rightarrow \frac{1}{c_2}\cdot\vec{a}_1 - \frac{c_3}{c_2}\vec{a}_3 = \vec{a}_2 \quad \text{if } c_2 \neq 0$$

$$A\vec{y} = \vec{b}$$

$$A\vec{x}_* = \vec{b} \quad \text{Known to me}$$

$$A\vec{x}_* = \vec{b} + \vec{0}$$

$$A\vec{x}_* + \vec{0} = \vec{b}$$

$$A\vec{x}_* + A\begin{bmatrix}-1\\ c_2\\ c_3\end{bmatrix} = \vec{b}$$

$$A\left(\vec{x}_* + \begin{bmatrix}-1\\ c_2\\ c_3\end{bmatrix}\right) = \vec{b}$$

Call $\vec{x}_* + \begin{bmatrix}-1\\ c_2\\ c_3\end{bmatrix} = \vec{y}$ $\quad \vec{x}_* - \vec{y} = \begin{bmatrix}1\\ -c_2\\ -c_3\end{bmatrix}$

Is $\vec{y} = \vec{x}_*$? ~~NO! Because~~

~~$\begin{bmatrix}-1\\ c_2\\ c_3\end{bmatrix} \neq \vec{0}$~~

~~If $\vec{x}_* = \vec{y}$~~

~~$\Rightarrow \vec{x}_* - \vec{y} = \vec{0}$~~

If $\vec{x}_*$ is a solution

then $\vec{y}$ is also a solution

In real numbers if

$a \cdot b = 0$ $\qquad$ if $a \in \mathbb{R}$, $b \in \mathbb{R}$.

Matrices: $A \cdot \vec{b} = 0$

$A \neq 0$

$\vec{b} \neq 0$

Consider: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3.14 \\ 3.14 \end{bmatrix}, \begin{bmatrix} 16 \\ 16 \end{bmatrix}, \begin{bmatrix} 513 \\ 513 \end{bmatrix} \right\}$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$