EECS 16A Lecture 5.

Today:

- Proofs continued
- Matrices as linear transformations

- Tech Survey: Please fill out
- Study groups survey.
- Watch Fa2019 lectures
- Wednesday HW Party.

Thm: If the columns of A are linearly dependant, then $A\vec{x} = \vec{b}$ does not have a unique solution.

Last time: 3x3 case.

n×n case is identical.

Known:

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

n columns. Rows don't matter

Columns are linearly dependant.

So some column $\vec{a}_k$

$$\vec{a}_k = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{k-1}\vec{a}_{k-1} + c_{k+1}\vec{a}_{k+1} + \cdots + c_n\vec{a}_n \quad (*)$$

→ Defn of linear dependance.

→ We want to connect this to the matrix A, and solutions to $A\vec{x} = \vec{b}$.

How can we go from information about individual columns to information about A?

Can we rewrite the equation (*) in terms of A?

Want:

$$\boxed{A\vec{x} = \vec{b} \text{ does not have a unique solution.}}$$

i.e. either there is no solution or there are ≥ 2 solutions.

If possible let $\vec{x}_*$ be a unique solution. We will show this is not possible.

To write a column $\vec{a}_k$ in terms of A,

we can say

$$\underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{kth position} = \vec{a}_k$$

$$\begin{bmatrix} \vec{a}_1, & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{a}_1$$

$$\boxed{\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{a}_1 c_1 + \cdots + \vec{a}_n c_n}$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{bmatrix} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{k-1}\vec{a}_{k-1} + c_{k+1}\vec{a}_{k+1} + \cdots + c_n\vec{a}_n$$

0 → kth position

no $\vec{a}_k$

So we can rewrite (*) as

$$A \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ 0 \\ \vdots \\ c_n \end{bmatrix} \longrightarrow A\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \right) = 0$$

or equivalently

$$A \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \\ -1 \\ c_{k+1} \\ \vdots \\ c_n \end{bmatrix}}_{\text{call this vector } \vec{w}.} = 0$$

-1 ← kth position

Atleast one entry of $\vec{w}$ is non-zero. So $\vec{w} \neq \vec{0}$.

Is $\vec{w} = \vec{0}$? → NO.

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now: $A \cdot \vec{w} = \vec{0}$. (**)

But $\vec{w}$ is not the zero vector.

$\vec{w}$ has a non zero entry at the kth position.

A is not a zero matrix.

Interesting....

How can this help us? We want to show that if $\vec{x}_*$ is a solution, there must be another solution as well.

We know: $A\vec{x}_* = \vec{b}$.

$$\Rightarrow \quad A\vec{x}_* + \vec{0} = \vec{b} \qquad \text{Eq } (**)$$

$$\Rightarrow \quad A\vec{x}_* + A\vec{w} = \vec{b}$$

$$\Rightarrow \quad A(\vec{x}_* + \vec{w}) = \vec{b}. \qquad \vec{x}_* + \vec{w} + \vec{x}_*$$

$$\Rightarrow \quad \vec{x}_* + \vec{w} \text{ is also a solution!}$$

Now $\vec{x}_*$ is not the only solution!

We can say.
$\vec{x}_* + \vec{w} + \vec{w} + \vec{w} + \vec{w} \ldots$ is a sol^n

$\boxtimes$ QED
Quad et demonstrodum.

So if the columns of A corresponding to your tomography machine are linearly dependant, then you cannot recover an accurate image.

Thm: If $A\vec{x} = \vec{b}$ has two or more solutions, then the columns of A are linearly dependant.

Note: This is related to the previous theorem, but it is very different!

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Proof:

Known:

$A\vec{x} = \vec{b}$ has two distinct solutions, say $\vec{x}_1$ and $\vec{x}_2$.

$\vec{x}_1 \neq \vec{x}_2$

$\Rightarrow$ $\boxed{\begin{aligned} A\vec{x}_1 &= \vec{b} \\ A\vec{x}_2 &= \vec{b} \end{aligned}}$

$A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}$

$\boxed{A\vec{x}_1 - A\vec{x}_2 = \vec{0}.}$

$A(\vec{x}_1 - \vec{x}_2) = \vec{0}.$

To show:

Columns of A are linearly dependant.

Mathematically some $\vec{a}_k$

$\vec{a}_k = \boxed{c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n}$

no $\vec{a}_k$

Let $\vec{x}_1 - \vec{x}_2 = \vec{y}$.

Let $\vec{y}$ have entries $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

$\boxed{\vec{y} \neq \vec{0}}$

$\boxed{A\vec{y} = 0}$ Can this give me information about the columns of A?

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \vec{0}$$

$y_1, y_2 \ldots y_n \in \mathbb{R}$ scalars.

$$\Rightarrow y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n = \vec{0}$$

Because $\vec{y} \neq 0$, atleast one of the entries is not equal to zero. Let us call this y_ℓ.

$$\Rightarrow y_\ell \vec{a}_\ell = -y_1\vec{a}_1 - \cdots - y_{\ell-1}\vec{a}_{\ell-1} - y_{\ell+1}\vec{a}_{\ell+1} \cdots - y_n\vec{a}_n$$

Because $y_\ell \neq 0$

$$\Rightarrow \vec{a}_\ell = \frac{-y_1}{y_\ell}\vec{a}_1 + \cdots \frac{-y_n}{y_\ell}\vec{a}_n$$

$\rightarrow \vec{a}_\ell$ is a lin comb of other columns. ■ QED.

Linear dependance   Alternate definition.

$\{\vec{a}_1, \vec{a}_2 \cdots \vec{a}_n\}$ are linearly dependant if there exist constants $c_1, c_2 \cdots c_n$ not all zero such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = 0.$$

$+c_i\vec{a}_i$

Equivalent to the other definition.

If $c_i \neq 0$, then:

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots c_{i-1}\vec{a}_{i-1} + c_{i+1}\vec{a}_{i+1} + \cdots + c_n\vec{a}_n = -c_i\vec{a}_i$$

$$\Rightarrow \frac{-c_1}{c_i}\vec{a}_1 - \frac{c_2}{c_i}\vec{a}_2 - \cdots - \frac{c_n}{c_i}\vec{a}_n = \vec{a}_i$$

i.e. $\vec{a}_i$ is represented as a linear combination of other columns.

$\{\vec{a}_1, \vec{a}_2, \vec{0}\}$

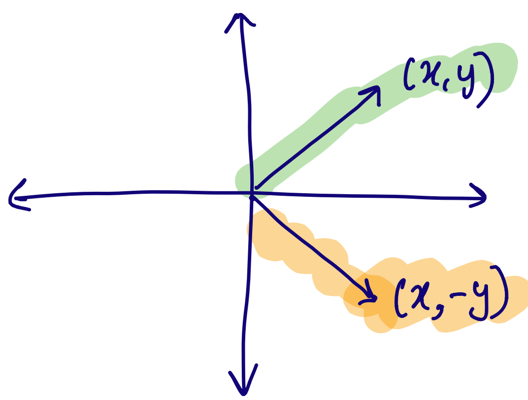
① Def ①: Write any column as a linar combination of other columns.

$$\vec{0} = 0\cdot\vec{a}_1 + 0\cdot\vec{a}_2$$

Matrix - vector multiplication is useful for representing systems of equations. But more broadly, matrices are "operators" that transform a vector into another vector.

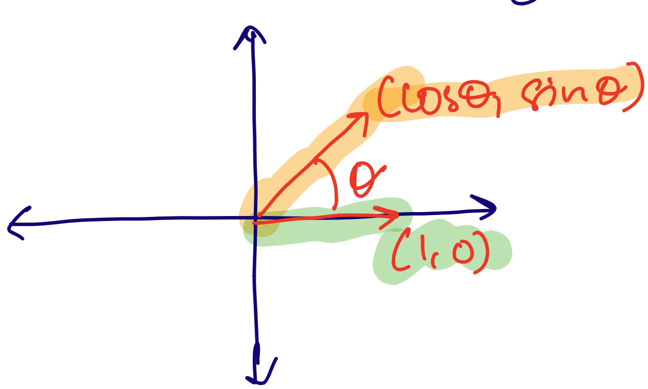
→ Discussion section (more examples)

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ Reflection matrix.



In particular, matrices are linear transformations.

e.g. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



Linear transformation

f: is a linear transformation if:

$$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$$

$$f(\alpha \cdot \vec{x}) = \alpha \cdot f(\vec{x}) \qquad \alpha \in \mathbb{R} \text{ (scalar)}$$

e.g. $f(\vec{x}) = 2\vec{x}$ is linear.

$f(x) = x^2$ is not linear.

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

$A(\alpha \cdot \vec{x}) = \alpha \cdot (A\vec{x})$

Check: Do matrix-vector mult. satisfy?

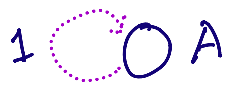
Vectors are used to represent "state" of a system

eg. the "state" of a car $\vec{s} = \begin{bmatrix} x \\ y \\ v \end{bmatrix}$ $\rightarrow$ x position, y position, velocity.

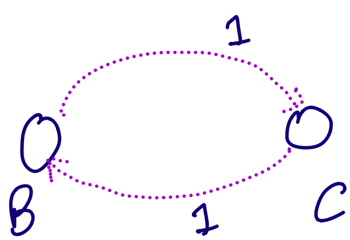
If this is changing with time

$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix}$$

A system of reservoirs and pumps.



A, B, C are tanks of water



What is the "state" of such a system?

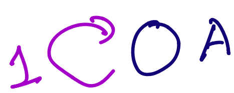
$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$x_A(t) \rightarrow$ Water in A at time t.

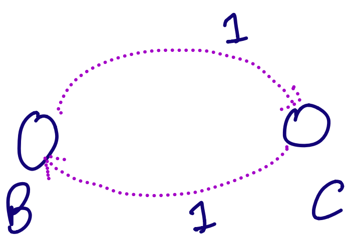
$$\vec{x}(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Say we interconnect these using some pumps.

Pumps run every time the clock ticks (e.g. every second)



: Every time the pump runs all water from A moves back into A.



: Everytime the pump runs all water, fraction(1) from B goes into C and all water (fraction 1) from C goes into B.

How can I represent this mathematically?

$$\left.\begin{aligned} x_A(t+1) &= x_A(t). \\ x_B(t+1) &= x_C(t) \\ x_C(t+1) &= x_B(t). \end{aligned}\right]$$

System of equations that describes evolution of the state.

Write in matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}}_{\vec{x}(t+1)}$$

$$\boxed{Q \cdot \vec{x}(t) = \vec{x}(t+1)}$$

What happens when we run the pumps twice? Guess?

$$\boxed{Q\cdot}\,\vec{x}(t+1) = \underline{\vec{x}(t+2)}$$

$$\boxed{Q\cdot(Q\cdot}\,\boxed{\vec{x}(t))} = \boxed{\vec{x}(t+2)}.$$

What is $\boxed{Q\cdot Q}$?

Matrix - Matrix multiplication

2x2

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \underbrace{\begin{matrix} b_{11} \\ b_{21} \end{matrix}}_{\vec{b_1}} & \underbrace{\begin{matrix} b_{12} \\ b_{22} \end{matrix}}_{\vec{b_2}} \end{bmatrix}}_{B}$$

Two vectors stacked.

$$= \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix}$$

$A \cdot \vec{b_1}$ = vector

$A\vec{b_2}$ = vector.

$$= \begin{bmatrix} A\vec{b_1} & A\vec{b_2} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{12}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{array}\right]$$

2x2 matrix.

2 stacked vectors.

In general: $A \cdot B$

$$= A \cdot \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \\ | & | & & | \end{bmatrix}$$

Go back to example..