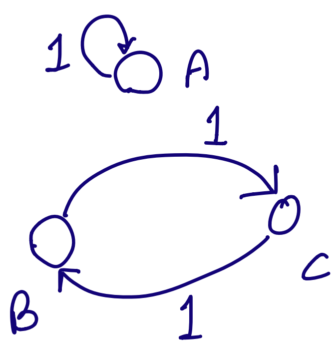
EECS 16A   Lecture 6

- Feedback survey (thanks!)
  - → Audio Video fine
  - → Speed - most are happy
  - → Some want more questions / others want fewer.
- Lab
- Proctoring HW.

Today
- More pumps
- Matrix - matrix multiplication
- Matrix inversion $f(x) = 2x, \ g(x) = \frac{1}{2}x.$

From last time



Transition matrix $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

$$\vec{x}(t+1) = Q \cdot \vec{x}(t).$$

$$\vec{x}(t+2) = Q \cdot \vec{x}(t+1)$$

substitute.

$$\vec{x}(t+2) = Q \cdot \underbrace{Q \cdot \vec{x}(t)}.$$ → Matrix vector mult.

Can we understand $Q \cdot Q$?

First: Just by thinking it through, what happens when pumps run twice?

What about $Q \cdot Q \cdot Q \cdots$???

→ When we operate twice - we get back to our original state!

Matrix-matrix multiplication

2x2 case

$$\vec{b_1} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \qquad \vec{b_2} = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$A \cdot \vec{b_1} =$ vector

$A \cdot \vec{b_2} =$ vector

$$= A \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix}$$

$$= \begin{bmatrix} A\vec{b_1} & A\vec{b_2} \end{bmatrix} = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} \cdot b_{22} \end{bmatrix}$$

2x2 matrix

"Stacking of vectors"

General:

$$A \cdot B = A \begin{bmatrix} \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ A\vec{b_1} & A\vec{b_2} & \cdots & A\vec{b_n} \\ | & | & & | \end{bmatrix}$$

① Matrix mult is not commutative.

$$AB \neq BA$$

② Matrix mult is associative

$$A(B \cdot C) = (AB) \cdot C$$

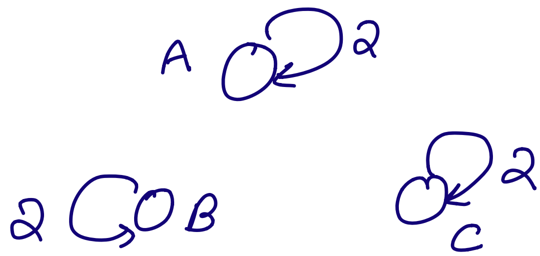
$$Q^2 = Q \cdot Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ "Identity matrix"

Identity function: $f(x) = x$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Another example



$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

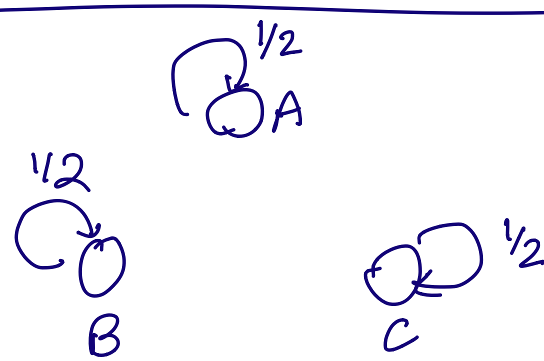
"non-conservative system."

$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

$Q \cdot Q \cdot Q \cdots$

2^{10}

Example 3



$$R = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

"non-conservative system"

$R \cdot R \cdots \quad R \cdots$

$$\vec{x}(t+1) = R \cdot \vec{x}(t)$$

What if I run Q, then R?

$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

$$\vec{x}(t+2) = R \cdot \vec{x}(t+1)$$

$$= R \cdot Q \cdot \vec{x}(t)$$

$$= \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}(t)$$

R is the inverse matrix of Q

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}(t)$$

$$Q \cdot R = R \cdot Q = I$$

<u>Inverses</u>: $f(x) = 2x$, $g(x) = \frac{1}{2}(x)$

$$g(f(x)) = x$$

g is the inverse of f.

$f(x) = 0 \cdot x \longrightarrow$ invertible? NO.

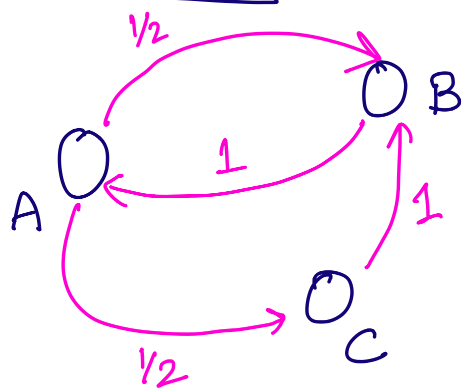
<u>Definition</u>: P, Q be square matrices.

Matrix P is said to be the inverse of matrix Q (and vice versa) if $P \cdot Q = Q \cdot P = I$.

<u>Notation</u>: $P = Q^{-1}$ "Q-inverse"

$$Q \cdot Q^{-1} = Q^{-1} \cdot Q = I$$

Example ④



$$x_A(t+1) = 0 + 1 \cdot x_B(t) + 0$$
$$x_B(t+1) = \tfrac{1}{2} x_A(t) + 0 + 1 x_C(t)$$
$$x_C(t+1) = \tfrac{1}{2} x_A(t) + 0 + 0$$

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \tfrac{1}{2} & 0 & 1 \\ \tfrac{1}{2} & 0 & 0 \end{bmatrix}}_{Q} \vec{x}(t)$$

$$\vec{x}(t+1) = \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}$$

out flow of A

outflow from B

$$Q \cdot Q = Q^2$$

$$\begin{bmatrix} 0 & 1 & 0 \\ \tfrac{1}{2} & 0 & 1 \\ \tfrac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \tfrac{1}{2} & 0 & 1 \\ \tfrac{1}{2} & 0 & 0 \end{bmatrix}$$

$$Q^{100} = \begin{bmatrix} .4 & .4 & .4 \\ .4 & .4 & .4 \\ .2 & .2 & .2 \end{bmatrix}$$

Inverse? Want P such that:

$$\vec{x}(t) = P \cdot \vec{x}(t+1)$$
$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

$$= \underbrace{Q \cdot P}\cdot \vec{x}(t+1)$$

$$\boxed{Q \cdot P = I}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3 \qquad\qquad \vec{c}_1 \quad \vec{c}_2 \quad \vec{c}_3$$

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\boxed{A\vec{x} = \vec{b}}$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 \end{array}\right]$$

Gaussian Elimination

steps do not care about $\vec{b}$!

↳ RHS of the augmented matrix.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_1]{\text{Swap}} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \times 2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \times (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \overset{P_{11}}{0} & \overset{P_{12}}{0} & \overset{P_{13}}{2} \\ 0 & 1 & 0 & \overset{P_{21}}{1} & \overset{P_{22}}{0} & \overset{P_{23}}{0} \\ 0 & 0 & 1 & \overset{P_{31}}{0} & \overset{P_{32}}{1} & \overset{P_{33}}{-1} \end{array}\right]$$

$$\begin{array}{ccc} \vec{P_1} & \vec{P_2} & \vec{P_3} \end{array}$$

$$P = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$Q \cdot P = I$$

$\boxed{\text{HW. Check this}}$

Inversion $\longleftrightarrow$ Systems of linear eqns $\longleftrightarrow$ Linear dependance

Thm: If the columns of matrix A are linearly dependant then Matrix A is not invertible (A^{-1} does not exist)

$P \Longrightarrow q$

raining $\Longrightarrow$ clouds.

Not q $\Longrightarrow$ Not p

If no clouds $\Longrightarrow$ cannot have rain.

If matrix A is invertible $\Longrightarrow$ then, the columns of matrix A are linearly independant