EECS 16A Lecture 7

Logistics

- CSM sections (Small group tutoring)
- Study groups
  - ↳ be inclusive
    - No question is a bad question.
- HW3 is due
  - Attend office hours + discussion.
  - Stay on top of HW.
- FA19

How to find inverse?

→ Gaussian Elimination.

Today:

$\boxed{f(x) = 2x \to \text{invertible.} \qquad g(x) = 0 \cdot x}$

① **Thm:** If the columns of square matrix A are linearly dependant then matrix A is not invertible.

(If square matrix A is invertible, then the columns of A are linearly independant)

② **Thm:** If A is an invertible matrix, then $A\vec{x} = \vec{b}$ has a unique solution.

③ **Nullspace**

If A is invertible the $\{\vec{0}\}$ is the only vector in the Nullspace of A.

④ **Vector space**

Thm: If the columns of A are linearly dependant, then matrix A is not invertible.

Proof:

Known/ Beginning:

$$A = \begin{bmatrix} \vec{a_1}, \vec{a_2} \cdots \vec{a_n} \end{bmatrix}$$

Note 3

By the definition of lineardep: there exists

$c_1\vec{a_1} + c_2\vec{a_2} + \cdots + c_n\vec{a_n} = \vec{0}$
and not all c_i's are equal to 0.

- If possible let A^{-1} exist.

$$A^{-1} \cdot A = A \cdot A^{-1} = I.$$

Rewrite in matrix -vector form

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

$$A \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \quad (*)$$

To show:

A^{-1} does not exist.

→ Try to do a proof by contradiction!

Assume, that A^{-1} does exist.

Multiply both sides by A^{-1}

$$A^{-1}\left(A \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right) = A^{-1} \cdot \vec{0}$$

$$I \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

This implies all c_i's are equal to zero!

But this is a contradiction!

$\Longrightarrow$ A^{-1} cannot exist!

QED.

$A\vec{x} = \vec{b} \rightarrow$ infinite solutions.

Thm: If A is an invertible matrix, then $A\vec{x} = \vec{b}$ has a unique solution.

Proof:

Known: A is invertible

A^{-1} exists. $A^{-1}A = A \cdot A^{-1} = I$

To show:

1. There exists atleast one solution
2. Any other sol^n is equal to the first.

$\rightarrow$ There cannot be 2 distinct solutions

Solutions for $A\vec{x} = \vec{b}$

Guess: $\vec{x}_0 = A^{-1} \cdot \vec{b}$

Check: $A\vec{x}_0 = A\,(A^{-1} \cdot \vec{b})$

$$= A \cdot A^{-1} \cdot \vec{b}$$

$$= I \cdot \vec{b}$$

$$= \vec{b} \quad !$$

$\vec{x}_0$ is a solution!

Aside

Now if possible let $\vec{x}_1 \neq \vec{x}_0$ be another solution.

$$\Longrightarrow \quad A\vec{x}_1 = \vec{b}$$

$$\longrightarrow \quad (A\vec{x})A^{-1} = \vec{b} \cdot A^{-1}$$

Multiply both sides by A^{-1}

$$\underbrace{A^{-1}(A}_{I}\,\vec{x}_1) = A^{-1}\vec{b}$$

$$\vec{x}_1 = A^{-1}\vec{b}$$

$$= \vec{x}_0 \;!!$$

Contradiction!

Aside: $2x = 4$

Question: $0x = 4$

Guess: Consider $x_0 = 2^{-1} \cdot 4$

$2^{-1} = 1/2.$

Check: $2 \cdot x_0 = 4$

$\Rightarrow 2 \cdot (2^{-1} \cdot 4) = 4$

$\Rightarrow 2 \cdot 2^{-1} \cdot 4 = 4$

$\Rightarrow 1 \cdot 4 = 4$

$\Rightarrow 4 = 4.$

Therefore x_0 satisfies my eqn.

A is an invertible matrix $\Longleftrightarrow$ $A\vec{x} = \vec{b}$ has a unique solution.

$\Longleftrightarrow$ A has linearly independent columns.

① Induction CS70

② SVD 16B.

$\Longrightarrow$ A has a trivial nullspace.

$A\vec{x} = \vec{b}$

$$A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ How many solutions does it have?

→ A is invertible : unique solution?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sol^n: $\vec{x}$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → The unique solution!

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ +1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ Gaussian elimination

$x_2 = t$

$x_1 + 2x_2 = 0 \implies x_1 + 2t = 0$

$\implies x_1 = -2t$

$\vec{v} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} \leftarrow$ a solution to the eq^n for all t.

$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} = \left\{ \vec{v} \,\middle|\, \vec{v} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad t \in \mathbb{R} \right\}.$

Span $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ ~~is a~~ are solutions to

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

"Nullspace" of matrix A.

Nullspace: set of all solutions to $A\vec{x} = \vec{0}$

Null(A), $N(A)$.

One example of a "Vector Space".

A is said to have a "trivial" nullspace" if $A\vec{x} = \vec{0}$ only has $\{\vec{0}\}$ as a solution.

$A\vec{x} = \vec{b} \longrightarrow A\vec{x}_1 = \vec{b}$ $\vec{x}_1$ is a solⁿ.
→ particular solution

$\vec{v}_0 \in \text{Null}(A)$ $A \cdot \vec{v}_0 = \vec{0}$ Homogeneous solution.

Then: $\vec{x}_1 + t \cdot \vec{v}_0$ is also a solution to $A\vec{x} = \vec{b}$ $t \in \mathbb{R}$

$$A(\vec{x}_1 + t \cdot \vec{v}_0) = \vec{b}$$

$$\Rightarrow A\vec{x}_1 + A \cdot t \cdot \vec{v}_0 = \vec{b}$$

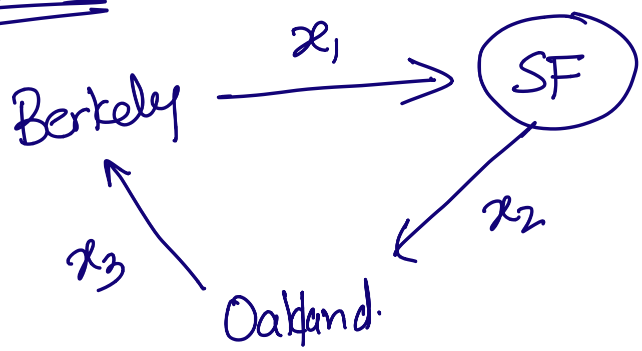
$$\Rightarrow A\vec{x}_1 + \underbrace{t \cdot A \cdot \vec{v}_0}_{\vec{0}} = \vec{b}$$

$$= A\vec{x}_1 \quad = \vec{b}$$

$A \cdot \vec{v}_0 = \vec{0}$

$A\vec{x}_1 = \vec{b}$

Traffic



$$x_1 - x_2 = 0$$
$$x_2 - x_3 = 0$$
$$x_3 - x_1 = 0$$

No accumulation of cars in cities

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nullspace!