EECS 16A Lecture 9

Logistics

- HW4 due Friday. (Feedback in Groups)
- HW5 last in scope for midterm
- Midterm Oct 5th.
- Roundtable (Virtual) today.
- Review Session Next week.

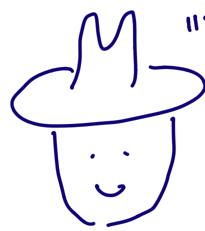
Last time .

- Determinants

$\longleftrightarrow$ Null spaces

$\longleftrightarrow$ Invertibility

Perspective: WHY is a formula Correct?



"Design Hat"

Today: Eigenvalues | Eigenspaces

Determinants $\longrightarrow$ 2x2 "Area" parallelogram

"Hypervolume"

Determinant $= 0 \iff$ Matrix is non-invertible

$\iff$ Nullspace is non-trivial.

$\iff$ columns are linearly dep.

$$\text{Det}\left(\begin{bmatrix} a & \alpha \cdot a \\ c & \alpha \cdot c \end{bmatrix}\right) = a(\alpha \cdot c) - \alpha a \cdot c$$

$$= \alpha ac - \alpha ac = 0.$$

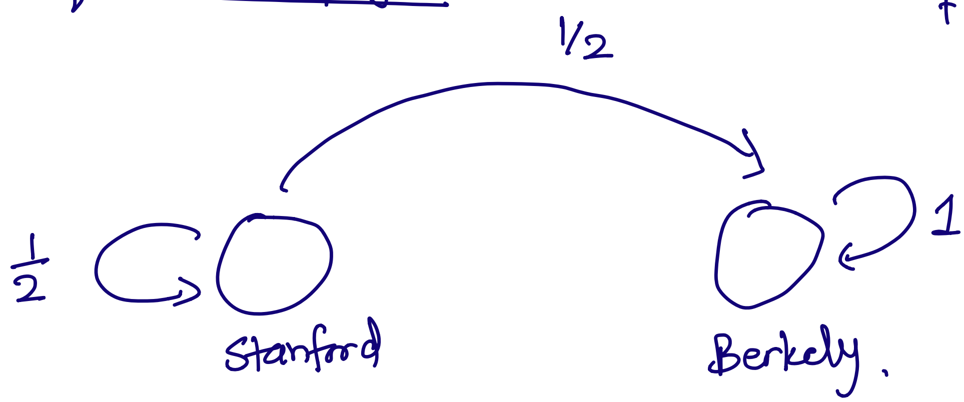
Go back Pumps

Google - 1 trillion dollars.

Google: Page Rank

Tale of two webpages

$$x_{stanf}[t+1] = \frac{1}{2} x_S[t] + 0 x_B(t)$$



$$\vec{x} = \begin{bmatrix} x_{stanford} \\ x_{berk} \end{bmatrix} \qquad Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}[1] = Q \cdot \vec{x}[0] = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{x}[2] = Q \cdot \vec{x}[1] = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)^2 \\ 1-\left(\frac{1}{2}\right)^2 \end{bmatrix}$$

$$\vec{x}[3] = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} = \begin{bmatrix} (1/2)^3 \\ 1-\left(\frac{1}{2}\right)^3 \end{bmatrix}$$

$$\vdots$$

$$\vec{x}[t] = \begin{bmatrix} (1/2)^t \\ 1-\left(\frac{1}{2}\right)^t \end{bmatrix} \qquad \lim_{t\to\infty} \vec{x}[t] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}[\infty] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What if instead $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Q \cdot \vec{x}[0] = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$ even when Q "transforms" it, it stays the same!

"Steady - state" of the system!

<u>In general:</u>

$$\vec{x}_{steady} = Q \cdot \vec{x}_{steady}$$

$$I \cdot \vec{x}_{steady} = Q \cdot \vec{x}_{steady}$$

$$\boxed{(Q - I)\, \vec{x}_{steady} = \vec{0}}$$

$$\Rightarrow Q \cdot \vec{x}_{steady} - I \vec{x}_{steady} = 0$$

$$(Q - I)\, \vec{x} = \vec{0}$$

A

We want $\vec{x} \in \text{Null}\,(Q-I)$!

$$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix} = Q - I$$

$$\downarrow$$

Null ( )

$$\left[\begin{array}{cc|c} -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 = 0.$$

x_2 is free.

$$\mathrm{Null}(Q-I) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$\boxed{= \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$$

"Eigenspace of matrix Q corresponding to eigenvalue 1"

Definition: Q be a square matrix.

$\lambda \in \mathbb{R}$ "Lambda"

If $\vec{x} \neq 0$ such that

$$\boxed{Q\vec{x} = \lambda \cdot \vec{x}}$$

then, we say that λ is an eigenvalue of Q

$\vec{x}$ is an eigenvector of Q.

And Null $(Q - \lambda I)$ is the eigenspace corresponding to e-value λ.

If $\lambda = 1$, then "steady state"

$\vec{x} \in$ "steady state"

Eigenvalues + Eigenspaces for $Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

Find $\lambda, \vec{x}$ such that

$$Q \cdot \vec{x} = \lambda \cdot \vec{x}$$

$$Q\vec{x} - \lambda \cdot I \circ \vec{x} = \vec{0}$$

$$(Q - \lambda I)\vec{x} = \vec{0}$$

Find $\vec{x} \in \underline{\mathrm{Null}(Q - \lambda I)}$.

$$Q - \lambda I = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1-\lambda \end{bmatrix}$$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

① Find λ. ✓

② Then we can find $\vec{x}$

$$\text{Det}(Q - \lambda I) = 0$$

$$\left(\tfrac{1}{2} - \lambda\right)(1-\lambda) - 0 \cdot \left(\tfrac{1}{2}\right)$$

$$= \left(\tfrac{1}{2} - \lambda\right)(1-\lambda) = 0$$

"Charachteristic polynomial"

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = \frac{1}{2}$ Eigenvalues

Trival null : $\{\vec{0}\}$.

Non-trivial : Null more vectors than just $\vec{0}$

$\lambda_1 = 1$, $\rightarrow$ we calculated that

span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ was the corresponding eigenspace.

$\lambda_2 = \frac{1}{2}$ $\qquad$ $\lambda_2 = \frac{1}{2}$

$$(Q - \lambda_2 I) = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Nullspace

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \xrightarrow{GE} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 + x_2 = 0$$

$\rightarrow x_2$

basic free

All vectors $\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \in \text{Null}(Q - \frac{1}{2}I)$.

$$\text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \in \underbrace{\text{Null}(Q - \tfrac{1}{2}I)}_{\text{eigenspace corresponding to } \frac{1}{2}}.$$

$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Q \cdot \vec{v} = \frac{1}{2}\vec{v}$

$Q \cdot Q \cdot \vec{v} = Q \cdot \frac{1}{2} \cdot \vec{v} = \left(\frac{1}{4}\right)\vec{v}$

$\underbrace{Q \cdot Q \cdots Q}_{t}\,\vec{v} = Q^t \cdot \vec{v} = \left(\frac{1}{2}\right)^t \cdot \vec{v}$

$\lambda_1 = 1$, $\quad \vec{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q \cdot \vec{v_1} = \vec{v_1}$

$\lambda_2 = \frac{1}{2} \quad \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad Q \cdot \vec{v_2} = \frac{1}{2}\vec{v_2}$.

$$\boxed{\vec{v}_3 = 2\vec{v}_1 + 3\cdot\vec{v}_2} \quad \underline{\text{(made up).}}$$

Not an e-vector!

$$Q\cdot\vec{v}_3 = Q\cdot\left(2\cdot\vec{v}_1 + 3\cdot\vec{v}_2\right)$$

$$= 2\cdot Q\cdot\vec{v}_1 + 3\cdot Q\cdot\vec{v}_2$$

$$= 2\cdot\vec{v}_1 + 3\cdot\tfrac{1}{2}\cdot\vec{v}_2$$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

HW:

$$\underline{\lambda_1 = 5}, \quad \underline{\lambda_2 = -1.}$$

$$\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A\cdot\vec{v}_1 = 5\,\vec{v}_1$$

- Matrices transfrom vectors.
- Some vectors are special
  - ↳ eigenvectors.
- $\lambda_1 = 1 \implies$ steady state

→ PageRank.

$A\vec{x} = \lambda\vec{x}$

$\implies$

$\underbrace{(A - \lambda I)}_{\text{nullspace}}\vec{x} = 0$

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

① Find λ_1, λ_2.

Consider $A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$

$\mathrm{Det}(A-\lambda I)$

$= (1-\lambda)(3-\lambda) - 4\times 2$

"Cha. Poly."

$= 3+\lambda^2-3\lambda-\lambda-8$

$= \lambda^2-4\lambda-5$

$= (\lambda-5)(\lambda+1)$

$\lambda_1 = 5, \quad \lambda_2 = -1.$

Now find eigenspace corr. to

$\lambda_1 = 5$

$\mathrm{Null}(A-5I)$

$A-5I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_1 basic $\quad$ x_2 free

$$x_1 - \frac{1}{2}x_2 = 0 \implies x_1 = \frac{1}{2}x_2$$

$$\vec{x} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} \quad \text{Null}(A-5I)$$

$$\text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$