EECS 16A

Module 1 Lecture 10

Logistics
- Midterm reminder 10/5
- Review : Thursday night.
- Most important: HW, discussion notes, labs.
- Read Piazza Posts!

We started the semester thinking about how to do imaging / tomography well.

Systems of equations, inversion, vector spaces

Today: How do eigenvalues help?

Properties of e-vals and e-vectors:

Thm: A be an $n \times n$ matrix.

$\lambda_1, \lambda_2, \cdots \lambda_n$ distinct eigenvalues.

$\lambda_i \neq \lambda_j$ for all i, j

$\lambda_1 \neq \lambda_2, \quad \lambda_1 \neq \lambda_3 \cdots \quad \lambda_1 \neq \lambda_n$

$\lambda_2 \neq \lambda_3 \cdots \quad \lambda_2 \neq \lambda_n$

$\vdots$

$\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n$ be the corresponding eigenvectors

$A \cdot \vec{v}_1 = \lambda_1 \vec{v}_1$

Then: $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n$ form a basis for $\mathbb{R}^n$.

$\mathbb{R}^2$, $n = 2$

Thm: A: 2×2 matrix.

λ_1, λ_2 , $\lambda_1 \neq \lambda_2$ eigenvalues.

$\vec{v_1}, \vec{v_2}$ are eigenvectors.

$\vec{v_1}, \vec{v_2}$ form a basis for $\mathbb{R}^2$.

Proof:

Known: $A \in \mathbb{R}^{2\times2}$

$A \cdot \vec{v_1} = \lambda_1 \vec{v_1}$

$A \cdot \vec{v_2} = \lambda_2 \cdot \vec{v_2}$

$\vec{v_1}, \vec{v_2} \neq \vec{0}$

$\lambda_1 \neq \lambda_2$

To show: $\vec{v_1}, \vec{v_2}$ form a basis.

Def of basis:

1. $\vec{v_1}, \vec{v_2}$ are linearly independant.
2. $\vec{v_1}, \vec{v_2}$ span all of $\mathbb{R}^2$.

Use a proof by contradiction.

If possible, let $\vec{v_1}$ and $\vec{v_2}$ be linearly dep.

$$\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} = \vec{0}$$

Not all α_i are equal to 0.

Rearrange

$$\vec{v_1} = -\frac{\alpha_2}{\alpha_1} \cdot \vec{v_2} \quad (*)$$

Say $\alpha_1 \neq 0$

(*) Multiply by A.

$$A \cdot \vec{v_1} = A \cdot \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot \vec{v_2}$$

$$= \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot A \cdot \vec{v_2}$$

$$\boxed{A \vec{v_1} = \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot \lambda_2 \cdot \vec{v_2}} \quad (**)$$

$$A \cdot \vec{v_1} = \lambda_1 \vec{v_1}$$

$$= \lambda_1 \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot \vec{v_2} \quad [\text{Using } (*)]$$

$$\boxed{A \vec{v_1} = \lambda_1 \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot \vec{v_2}} \quad (***)$$

$\Rightarrow$ $(**)$ and $(***)$ $\Rightarrow$

$$\left(-\frac{\alpha_2}{\alpha_1}\right) \lambda_2 \cdot \vec{v_2} = \lambda_1 \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot \vec{v_2}$$

$\Rightarrow$ $\lambda_2 = \lambda_1$!!! Contradiction!

Therefore: $\vec{v}_1$ and $\vec{v}_2$ must be linearly independant!

Now: $\vec{v}_1, \vec{v}_2$.

Question: Can I have a set of more than 2 vectors in a 2 dimensional space that are linearly independent?

$\Longrightarrow$ $\vec{v}_1$ and $\vec{v}_2$ must span $\mathbb{R}^2$.

$\rightarrow$ The must form a basis!

QED.

To show I can reach any $\vec{x} \in \mathbb{R}$, using $\vec{v}_1, \vec{v}_2$

$$\left[\underbrace{\left(\vec{v}_1 \quad \vec{v}_2\right)}\ \vec{x}\right] \leftarrow V = \{\vec{v}_1, \vec{v}_2\}$$

V is an invertible matrix $\Rightarrow$

$$\left[V \;\middle|\; \vec{x} \right] \text{ has a unique solution.}$$

Matrix transformations

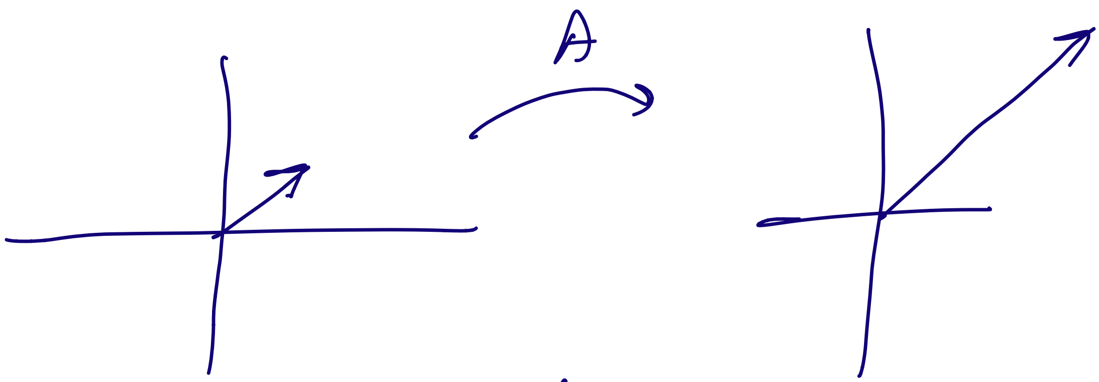
$$\vec{x}[t+1] = A \cdot \vec{x}[t]$$ "Dynamical system"

Steady state: $\vec{x} = A \cdot \vec{x}$

$\vec{x}$ is ~~ane~~ eigenvector with eigenvalue 1.

In general: $\vec{x}$ eigenvector with eigenvalue λ

$$A\vec{x} = \lambda \vec{x}$$



"Invariant"

A : 2×2 matrix 2 eigenvectors.

$\vec{x}$ that is not an eigenvector of A.

$A \cdot \vec{x}$

$\vec{v_1}, \vec{v_2} \cdots \vec{v_n}$ are the eigenvectors of A

$\lambda_1, \lambda_2 \cdots \lambda_n$ are e-vals (distinct).

We can write $\vec{x}$ as a linear combination of $\vec{v_1}, \vec{v_2} \cdots \vec{v_n}$!

$$\boxed{\vec{x} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n}}$$

$$\begin{aligned} A \cdot \vec{x} &= A(\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n}) \\ &= \alpha_1 A\vec{v_1} + \alpha_2 A\vec{v_2} + \cdots + \alpha_n A\vec{v_n} \\ &= \underbrace{\alpha_1 \cdot \lambda_1 \vec{v_1} + \alpha_2 \lambda_2 \vec{v_2} + \cdots + \alpha_n \cdot \lambda_n \vec{v_n}} \end{aligned}$$

$$\begin{aligned} A^2 \cdot \vec{x} &= A \cdot A\vec{x} \\ &= A(\qquad\qquad) \end{aligned}$$

$$= A\alpha_1 \lambda_1 \vec{u_1} + \cdots + A\cdot\alpha_n \lambda_n \vec{u_n}$$

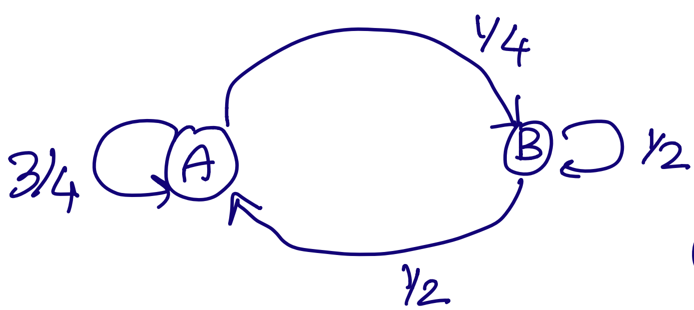
$$= \alpha_1 \lambda_1^2 \vec{u_1} + \cdots + \alpha_n \lambda_n^2 \vec{u_n}$$

$$A^t \cdot \vec{x} = \alpha_1 \lambda_1^t \vec{u_1} + \alpha_2 \lambda_2^t \vec{u_2} + \cdots + \alpha_n \lambda_n^t \vec{u_n}.$$

$\lambda_i = 1$

$\lambda_i < 1$

$\lambda_i > 1$



$$Q = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

$$Q - \lambda I = \begin{bmatrix} 3/4 - \lambda & 1/2 \\ 1/4 & \frac{1}{2} - \lambda \end{bmatrix}$$

E-vals, E-vectors of Q.

$$\det(Q - \lambda I) = \left(\frac{3}{4} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \frac{1}{4}\cdot\frac{1}{2}$$

$$= \frac{3}{8} + \lambda^2 - \frac{5}{4}\lambda - \frac{1}{8}$$

$$= \frac{1}{4} - \frac{5}{4}\lambda + \lambda^2 \leftarrow$$

$= (\lambda - \frac{1}{4})(\lambda - 1)$

$\longrightarrow$ HW. $\longrightarrow$ Practice

$\left.\begin{array}{l} \lambda_1 = 1, \quad \vec{u_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \lambda_2 = \frac{1}{4}, \quad \vec{u_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array}\right\}$ Compute $N(Q - \lambda I)$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow$ Basis for $\mathbb{R}^2$

Let $\vec{x}$ be any vector in $\mathbb{R}^2$

$$\vec{x} = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1, \quad \vec{u_1}$ $\qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda_2 = \frac{1}{4}, \quad \vec{u_2}$

$$\vec{x} = \alpha_1 \vec{u_1} + \alpha_2 \vec{u_2}$$

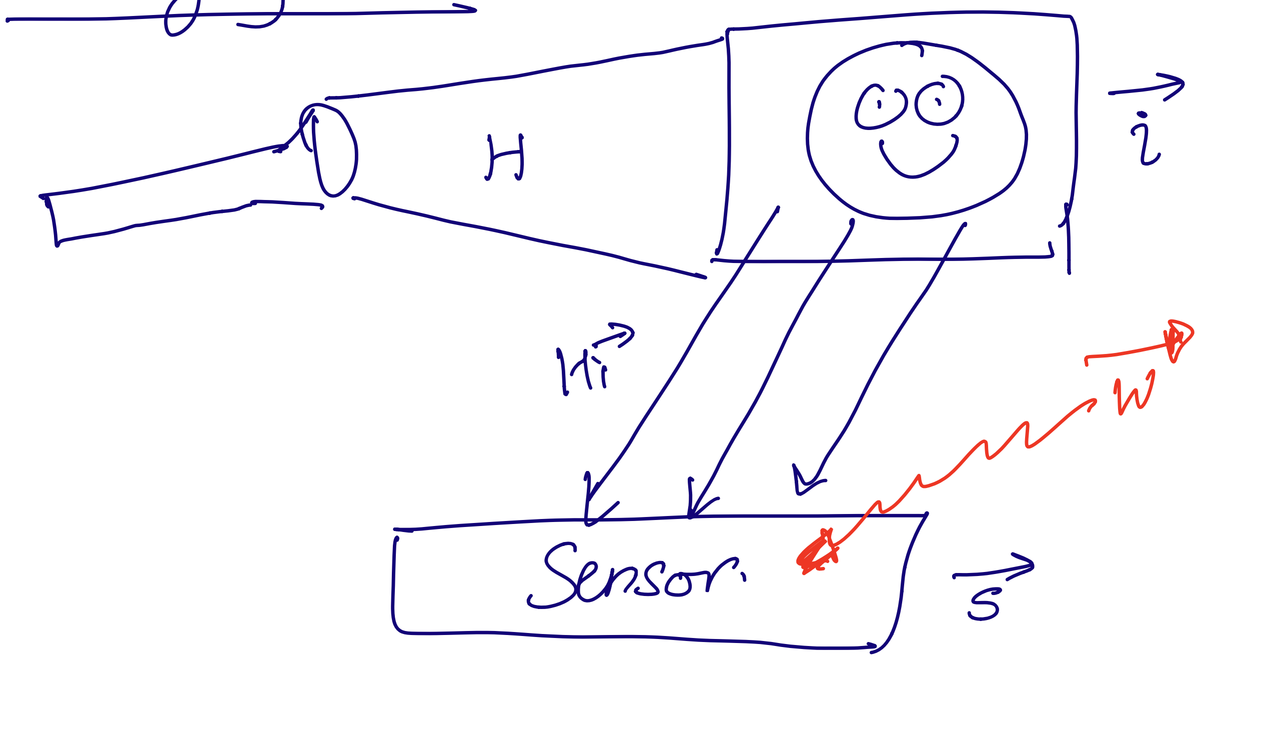
$$A\vec{x} = A\cdot\alpha_1\vec{v_1} + A\cdot\alpha_2\vec{v_2}$$

$$= \alpha_1\cdot 2\cdot\vec{u_1} + \alpha_2\cdot\frac{1}{4}\cdot\vec{u_2}$$

$$A^t\cdot\vec{x} = \alpha_1(2)^t\cdot\vec{u_1} + \alpha_2\left(\frac{1}{4}\right)^t\cdot\vec{v_2}$$

$$\lim_{t\to\infty} A^t\vec{x} = \alpha_1\vec{v_1} + 0$$

Imaging Lab



$$\underbrace{\vec{s}}_{\text{sensor}} = H \cdot \underbrace{\vec{i}}_{\text{image}}$$

H: measurement mask / matrix.

$$\vec{i} = H^{-1}\vec{s}$$

H^{-1} exists

$$\vec{s} = H\vec{i} + \vec{w}$$

$$H^{-1}\vec{s} = H^{-1}(H\vec{i} + \vec{w})$$

$$= \vec{i} + \underbrace{H^{-1}\vec{w}}_{\text{corruption}}$$

Eigenvectors of H^{-1}

$\vec{u}_1, \vec{u}_2 \dots \vec{u}_n$, $\lambda_1, \lambda_2 \dots \lambda_n$.

$$\vec{w} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_n \vec{u}_n$$

$$H^{-1}\vec{w} = \alpha_1 \lambda_1 \vec{u}_1 + \alpha_2 \lambda_2 \vec{u}_2 + \dots + \alpha_n \lambda_n \vec{u}_n$$

$$H^{-1} \quad \lambda \quad \longleftrightarrow \quad H \quad \frac{1}{\lambda}$$

Office Hours

Addendum: Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$.

$\vec{u}_1, \vec{u}_2$ are linearly indep.

Prove that: $\vec{u}_1, \vec{u}_2$ are a basis for $\mathbb{R}^2$.

$$V = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \qquad V \in \mathbb{R}^{2\times 2}$$

V has linearly indep columns.

V^{-1} exist.

We need to show that for any vector $\vec{x} \in \mathbb{R}^2$ we can write $\vec{x}$ as a linear combi of $\vec{u}_1$ and $\vec{u}_2$.

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \vec{x} \end{bmatrix}$$

show that such α_1, α_2 exist.

Consider: $V^{-1}\vec{x}$

Since V is invertible we know that this has a unique solution.

So α_1, α_2 exist.

So $\vec{x} \in \text{span}\{\vec{u_1}, \vec{u_2}\}$.

So $\{\vec{u_1}, \vec{u_2}\}$ is a basis for $\mathbb{R}^2$.
