EECS 16A Designing Information Devices and Systems I
Spring 2018

Midterm 2

Midterm 2 Solution

PRINT your student ID: ____

PRINT AND SIGN your name: ____________________, ____________________ ________________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ____

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

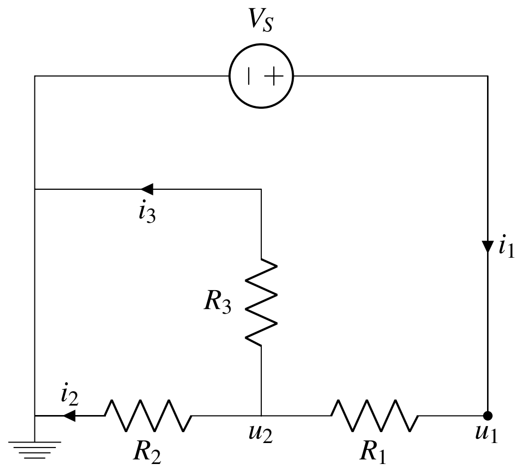
1. What do you enjoy most about EE16A? (1 Point)

2. What other courses are you taking this semester? (1 Point)

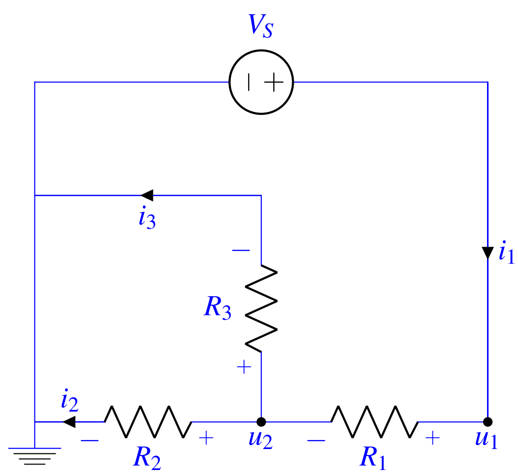
Do not turn this page until the proctor tells you to do so. You may work on the questions above.

3. Seven Steps of Highly Resistive Circuits (10 points)

(a) (3 point) Label the +/- polarity of voltage across each resistor element on the following circuit. Be sure to follow passive sign convention using the labeled currents.



Solution: Label polarity as shown.



(b) (7 points) Fill in the following matrix and unknown vector, such that they represent linearly independent equations for the circuit given above. Assume the resistor and voltage source values are known constants. **You do not need to solve this matrix.**

$$\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

Solution:

$$i_1 = i_2 + i_3$$

$$u_1 - u_2 = i_1 R_1$$

$$u_2 - 0 = i_2 R_2$$

$$u_2 - 0 = i_3 R_3$$

$$u_1 - 0 = V_s$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -R_1 & 0 & 0 & 1 & -1 \\ 0 & -R_2 & 0 & 0 & 1 \\ 0 & 0 & -R_3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_S \end{bmatrix}$$

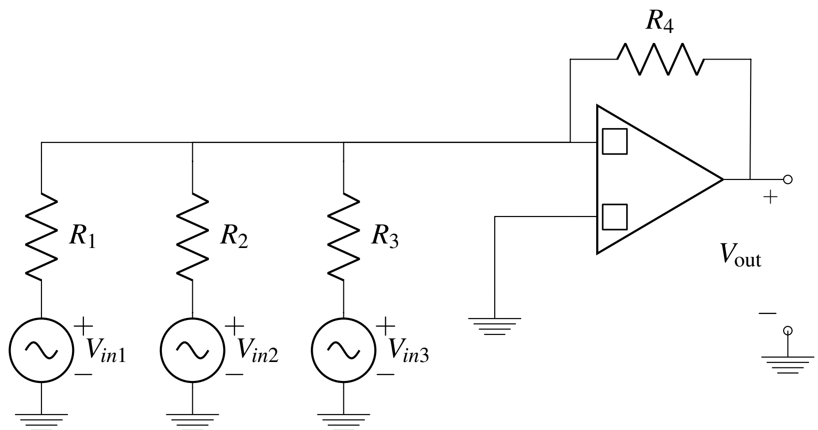
PRINT your name and student ID: ____

4. "Operational" Amplifiers (12 points)

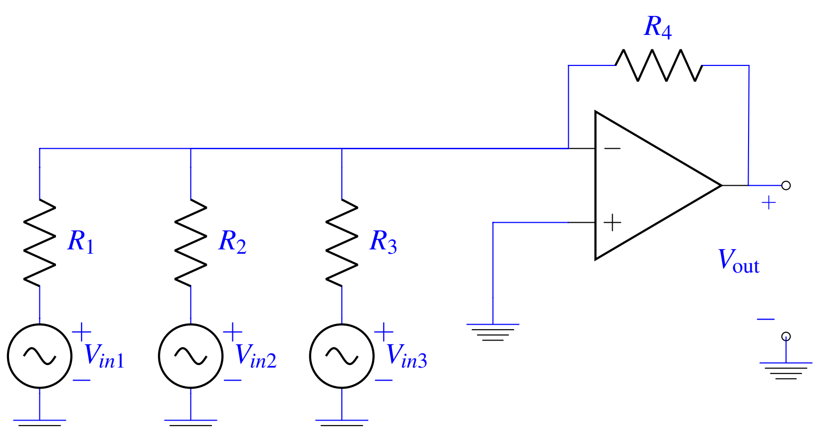
> "As an amplifier so connected can perform the mathematical operations of arithmetic and calculus on the voltages applied to its inputs, it is hereafter termed an 'operational amplifier'."
>
> John Ragazzini, Robert Randall and Frederick Russell
> Proceedings of IRE, Vol. 35, May 1947

In this problem we will explore some of the mathematical operations an op amp can perform.

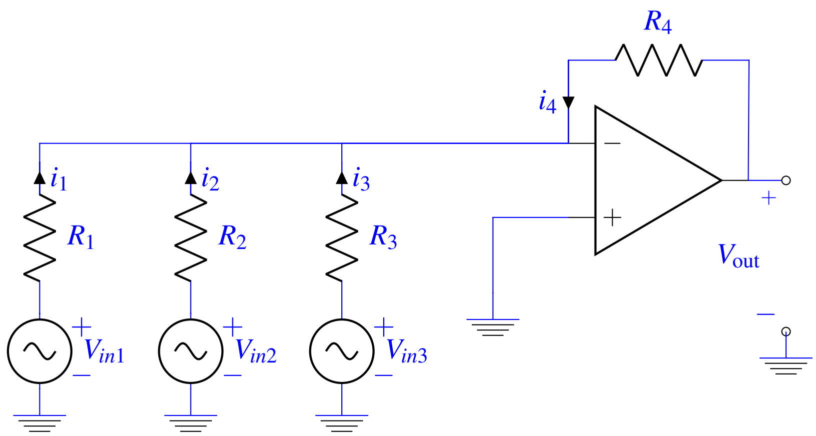
(a) (5 points)



i. Label the '$+$' and '$-$' terminals of the op amp above so that it is in negative feedback.
Solution:



ii. Derive an expression for V_{out} as a function of V_{in1},V_{in2}, and V_{in3}.
Solution:



Since no current enters the op amp, we get the following equation from KCL:

$$i_1 + i_2 + i_3 + i_4 = 0$$

By the Golden Rules $V_- = V_+ = 0V$. We use V_- to calculate the current in each branch:

$$i_1 = \frac{V_{in1}}{R_1} \qquad i_2 = \frac{V_{in2}}{R_2}$$

$$i_3 = \frac{V_{in3}}{R_3} \qquad i_4 = \frac{V_{\text{out}}}{R_4}$$

Putting this together with the KCL equation above yields

$$\frac{V_{in1}}{R_1} + \frac{V_{in2}}{R_2} + \frac{V_{in3}}{R_3} + \frac{V_{out}}{R_4} = 0$$

$$v_{out} = -(\frac{R_4}{R_1}V_{in1} + \frac{R_4}{R_2}V_{in2} + \frac{R_4}{R_3}V_{in3})$$

iii. Mark the operation below that is best represented by this configuration.

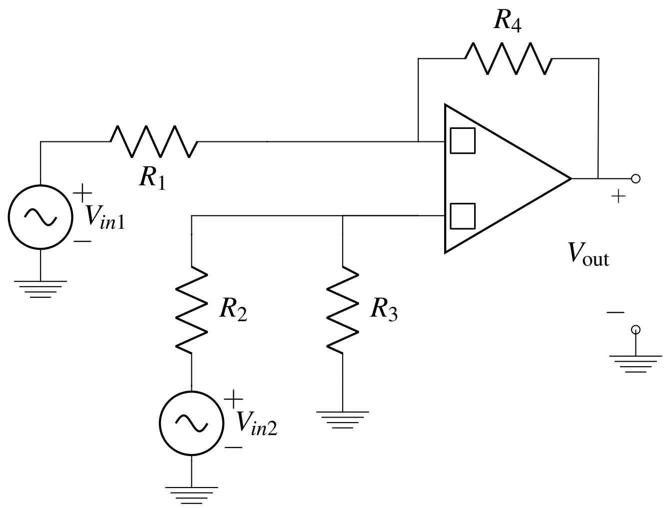
◯ **Subtraction** ◯ **Inner Product** ◯ **Differentiation** ◯ **Integration**

Solution: This operation is the **inner product** between the following two vectors:

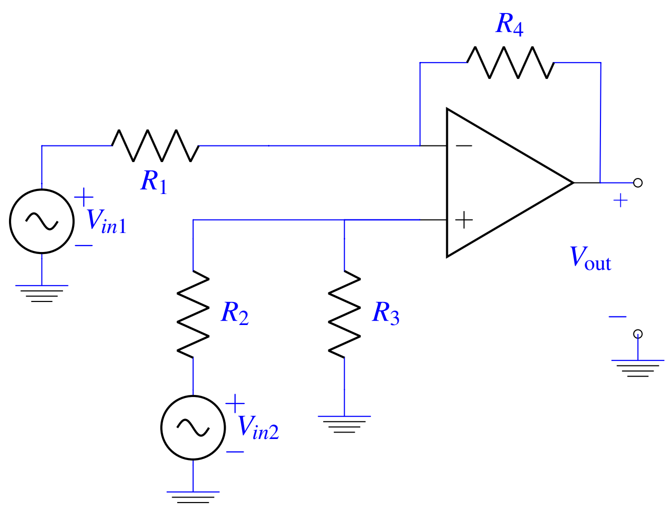
$$\begin{bmatrix} V_{in1} & V_{in2} & V_{in3} \end{bmatrix}^T$$

$$\begin{bmatrix} -\frac{R_4}{R_1} & -\frac{R_4}{R_2} & -\frac{R_4}{R_3} \end{bmatrix}^T$$

(b) (7 points)



i. Label the '+' and '−' terminals of the op amp above so that it is in negative feedback.
Solution:



ii. Derive an expression for V_{out} as a function of V_{in1} and V_{in2}.
Solution:
Using superposition:
Nulling V_{in1}, we see that V_+ is formed by a voltage divider:

$$V_+ = V_{in2}\left(\frac{R_3}{R_2+R_3}\right)$$

The output amplifies V_+ like a non-inverting amplifier:

$$V_{out2} = V_{in2}\left(\frac{R_3}{R_2+R_3}\right)\left(1+\frac{R_4}{R_1}\right)$$

Nulling V_{in2}, we see that V_+ must be 0 V. By the golden rules, $V_- = V_+$, and we can treat this like an inverting amplifier:

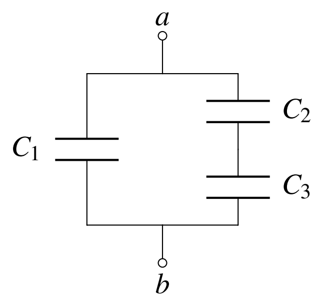
$$V_{out1} = -V_{in1}\left(\frac{R_4}{R_1}\right)$$

By superposition, we add the intermediate output voltages to determine the final output voltage:

$$V_{out} = V_{in2}\left(\frac{R_3}{R_2+R_3}\right)\left(1+\frac{R_4}{R_1}\right) - V_{in1}\left(\frac{R_4}{R_1}\right)$$

5. Equivalent Capacitance (9 points)

(a) (4 points) Find the equivalent capacitance between terminals a and b of the following circuit in terms of the given capacitors C_1, C_2, and C_3. Leave your answer in terms of the addition, subtraction, multiplication, and division operators **only**.



Solution:

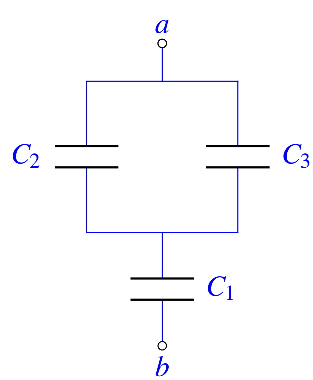
$$C_{eq} = C_1 + (C_2 || C_3)$$
$$C_{eq} = C_1 + \frac{C_2 C_3}{C_2 + C_3}$$

Here, $||$ represents the mathematical parallel operator ($a||b = \frac{ab}{a+b}$).

(b) (5 points) Find and draw a capacitive circuit using three capacitors, C_1, C_2, and C_3, that has equivalent capacitance of

$$\frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3}$$

Solution: This expression is the same as $C_1 || (C_2 + C_3)$, so C_2 and C_3 are in parallel with each other, and C_1 is series with both of them:



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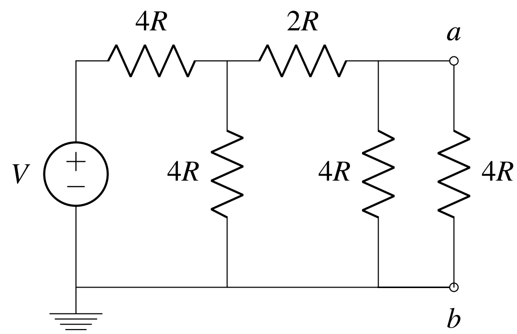
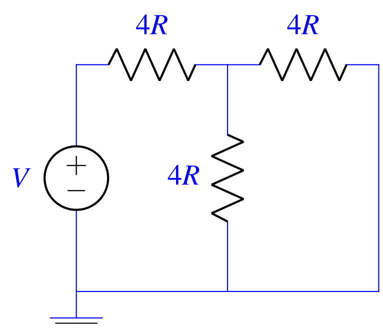
6. Power to Resist (6 points)

Find the power dissipated by the voltage source in the circuit below. Be sure to use passive sign convention.



Solution:

This circuit can be reduced using techniques similar to those used to analyze the R-2R ladder from homework. We want to find the equivalent resistance across the voltage source in Figure 6.2. Start by reducing the two resistors on the right to $4R \parallel 4R = 2R$. Then combine the other $2R$ resistor with this to get a new resistor of value $4R$ as in the circuit below.



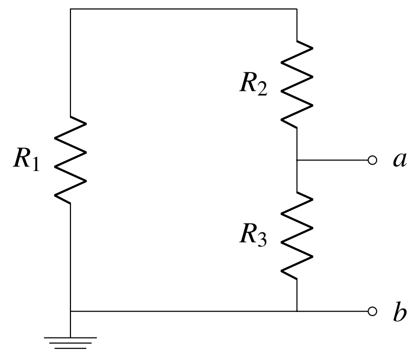
Once again we have $4R \parallel 4R = 2R$. This is finally in series with $4R$ giving us a total resistance of $4R + 2R = 6R$

$$P = VI = V\frac{-V}{6R} = -\frac{V^2}{6R}$$

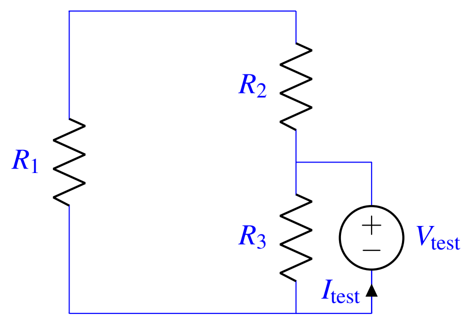
The negative sign is present because the voltage source actually provides power, which can also be seen by using passive sign convention.

7. Mechanical Thevenin and Norton Equivalents (10 points)

(a) (4 points) Find and draw the **Thevenin** equivalent circuit between terminals a and b in the circuit below. Clearly label the Thevenin equivalent voltage, V_{th}, and the Thevenin equivalent resistance R_{th}.



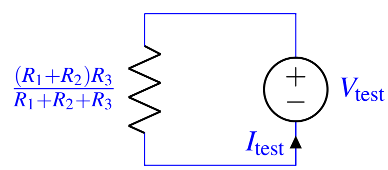
Solution: Start by finding V_{oc}, which is the voltage dropped across R_3. Since there are no sources in this circuit, the voltage across R_3, $V_{R_3} = 0$ and therefore $V_{\text{th}} = 0$. To find R_{th}, first null all independent sources (there are none). Then apply a test voltage to the terminals like so:



And calculate the current exiting the voltage source with Nodal Analysis or Resistor Equivalences. This solution uses resistive equivalences. R_{eq} as seen from the voltage source is

$$R_{\text{eq}} = (R_1 + R_2)||R_3 = \frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3}$$

Collapsing the resistors down into this R_{eq} turns the circuit into the one below:

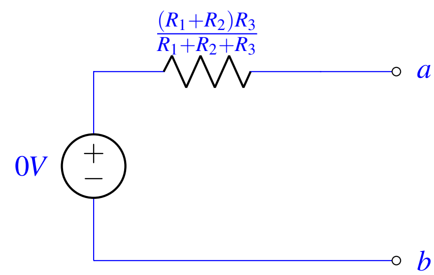


We can solve for the value of I_{test} with Ohm's Law.

$$V = IR \rightarrow I = \frac{V_{\text{test}}}{R_{eq}} = \frac{V_{\text{test}}}{\frac{(R_1+R_2)R_3}{R_1+R_2+R_3}} = V_{\text{test}} \frac{R_1 + R_2 + R_3}{(R_1 + R_2)R_3}$$

$$R_{th} = \frac{V_{\text{test}}}{I_{\text{test}}} = \frac{V_{\text{test}}}{V_{\text{test}} \frac{R_1+R_2+R_3}{(R_1+R_2)R_3}} = \frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3}$$

Therefore, the Thevenin Equivalent circuit is as follows:

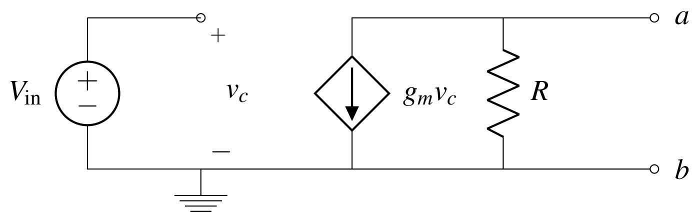


Circuits omitting the voltage source are also fine.

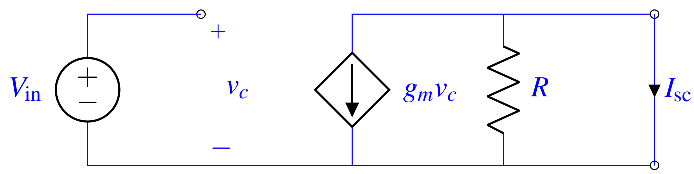
Common Mistakes:

- Calculating R_{th} by dividing V_{oc} by I_{sc}. This leads to a division by zero issue. The most reliable way to calculate R_{th} is using V_{test} / I_{test}, although in this problem you could also use equivalent resistances.
- Adding in a voltage source.
- Forgetting to take the reciprocal for parallel resistors. For parallel resistors $R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$.

(b) (6 points) Find and draw the **Norton** equivalent circuit between terminals a and b in the circuit below. Clearly label the Norton equivalent current, I_{no}, and the Norton equivalent resistance R_{no}.



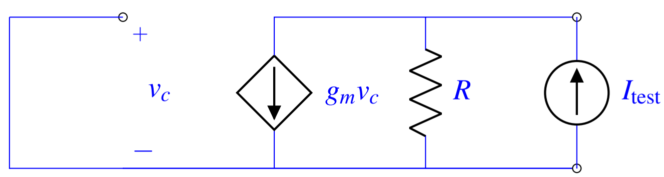
Solution: Begin by finding the short circuit current, I_{sc} which is equal to I_{no}, by shorting the terminals a and b.



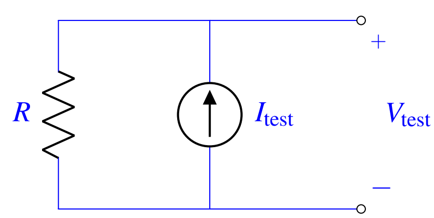
This sets the voltage across R to be 0V, since it is in parallel with an ideal wire. Therefore, the current from current source $g_m v_c$ will be what flows through the short, where $v_c = V_{\text{in}}$. Be careful, however: the current is flowing from the *negative* terminal to the *positive* terminal, so I_{sc} is actually $-g_m V_{\text{in}}$.

$$I_{\text{no}} = I_{\text{sc}} = -g_m V_{\text{in}}$$

Now, onto finding R_{no}. Null all independent sources, and apply a test current to the circuit as shown:



When the voltage source is nulled, $v_c = 0$, so the dependent current source $g_m v_c$ will output *zero* current and behave as an open circuit. So what is left is essentially a loop with the resistor R and I_{test}:



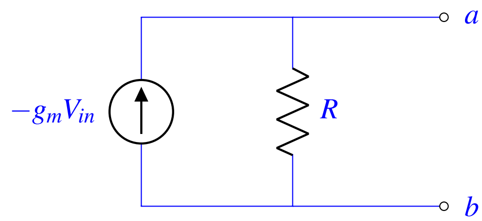
Because the resistor and current source are in a simple loop, the voltage across the resistor (and therefore V_{test}) can be calculated with Ohm's Law:

$$V_{\text{test}} = I_{\text{test}} R$$

And R_{no} can be calculated with:

$$R_{\text{no}} = \frac{V_{\text{test}}}{I_{\text{test}}} = \frac{I_{\text{test}} R}{I_{\text{test}}} = R$$

The Norton Equivalent circuit is as follows:



Common Mistakes:

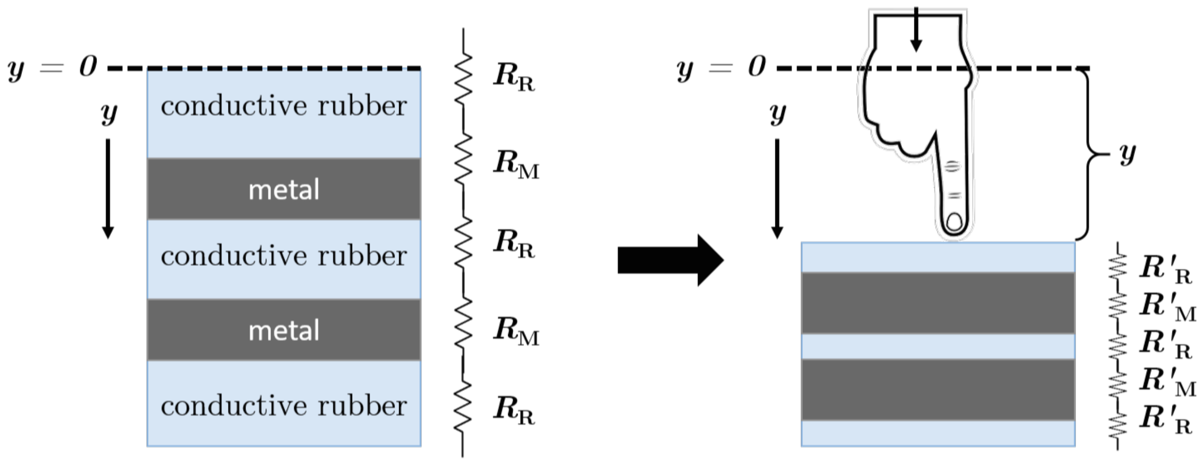
- Using the wrong direction/sign of the current source. A Norton equivalent circuit has the design above, so I_{no} should be negative.
- Using a dependent source instead of an independent source in the Norton equivalent circuit.
- Drawing a Thevenin equivalent circuit instead of a Norton equivalent circuit.
- Placing V_{test} / I_{test} in the wrong spot.
- Forgetting to cancel out the dependent source when doing V_{test} / I_{test}.

8. Midterms are a lot of Pressure (16 points)

In our labs, we used our resistive touchscreen to figure out where we were pressing in a 2D space. Now we'll use a new setup to determine how hard we're pressing! For this we'll use something called "pressure sensitive rubber," which incorporates conductive rubber and metal into one system. As the rubber is pressed, the conductive rubber portions are compressed, which changes the resistance. The metal plates do not change, but they assist in conduction through the material.

The pressure sensitive rubber system is shown below, with a resistive model next to the diagram. The resistivity of rubber and metal are represented by ρ_R and ρ_M respectively. When the system is at rest (no touch), the resistances of the rubber and metal are represented by R_R and R_M. The area of the sensor, as seen from above, is A.

To use the material, a finger presses on top of the system, compressing the rubber regions, creating a change in resistance, also shown below. Please answer the following questions related to the system.



Side View

Side View

(a) (2 point) Is the resistor model implementing resistors in series or parallel?

Solution: The resistors are in series.

(b) (3 points) If the values are $R_R = 1\,\text{k}\Omega$ and $R_M = 10\,\Omega$, what is the total resistance before pressing the system?

Solution: There are three R_R resistors and two R_M, so $R_{total} = 3\,\text{k}\Omega + 20\,\Omega = 3.02\,\text{k}\Omega$.

(c) (4 points) During the press, the length of each rubber portion is reduced by a factor of 5. (Its length is now 1/5 of its original value.) The size of the metal plates does not change. What is the new total resistance during a press?

Solution: If the length is reduced by a factor of 5, our R_R is reduced by a factor of 5:

$$R_R = \rho_R \frac{l}{A} \longrightarrow R'_R = \rho_R \frac{l/5}{A} = \frac{R_R}{5}$$

$$R'_{total} = \frac{3\,\text{k}\Omega}{5} + 20\,\Omega = 620\,\Omega$$

(d) (5 points) The force required to compress the rubber is $F = ky$, where k is a constant and y is the distance compressed (from the origin). Derive an expression for the resistance as a function of the pressing force F.

Write your answer in terms of the initial resistances (R_R and R_M), the resistivities (ρ_R and ρ_M), the area of the sensor, A, and the constant, k. Assume all rubber layers compress the same amount and uniformly.

Solution: First, let's consider the total resistance of just the rubber. If we press the rubber such that the rubber compresses by an amount y, this means we've reduced the length of our conductive rubber region to $l - y$, making our resistance of the rubber region:

$$\begin{aligned}
R_{R-press} &= \rho_R \frac{l-y}{A} \\
&= \rho_R \frac{l-F/k}{A} \\
&= \rho_R \frac{l}{A} - \rho_R \frac{F/k}{A}
\end{aligned}$$

The first term is equivalent to the initial (no press) resistance of one segment of rubber, $3R_R$.

$$R_{R-press} = 3R_R - \rho_R \frac{F}{kA}$$

To get the total resistance of the sensor, we can add the resistance of the metal potions. Since the metal portions do not change, their resistance is still $2R_M$.

$$R_{total}(F) = 3R_R - \rho_R \frac{F}{kA} + 2R_M$$

(e) (2 points) For a particular sensor, we find that the resistance is:

$$R(F) = \frac{8k\Omega \cdot mm^2}{A} - \left(100 \frac{\Omega \cdot m^2}{N}\right) \frac{F}{A}$$

We define the sensitivity of the sensor, S, to be the change in resistance per unit of force:

$$S = \left| \frac{dR}{dF} \right|$$

If we want to increase sensitivity, how should we change the area of the sensor? Justify your answer in 1-2 sentences.

Solution: We can calculate the sensitivity:

$$S = \left| \frac{d}{dF} \left(\frac{8k\Omega \cdot mm^2}{A} - \left(100 \frac{\Omega \cdot m^2}{N}\right) \frac{F}{A} \right) \right| = 100 \frac{\Omega \cdot m^2}{N} \left| \frac{1}{A} \right|$$

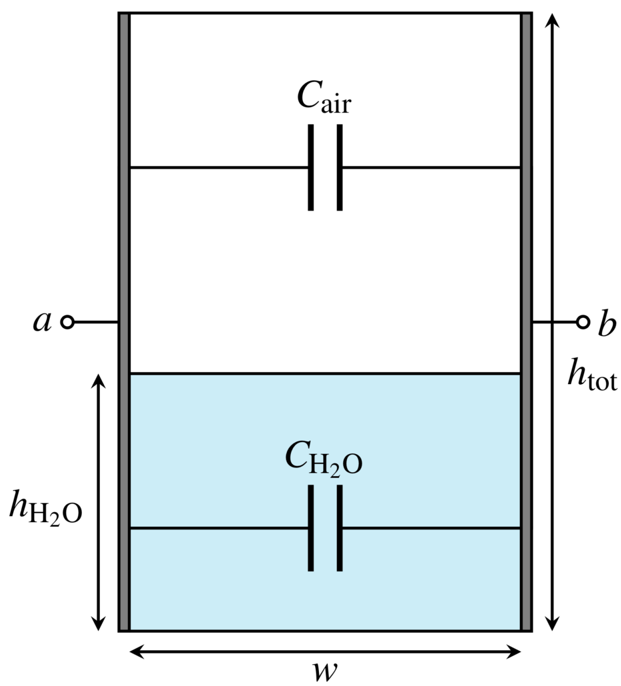
Sensitivity is inversely proportional to A, so we should **decrease** the area of the sensor if we want to increase sensitivity.

Common Mistakes:

- Correctly saying "decrease A" but for the wrong reasons. Decreasing A will increase R and $\frac{dR}{dF}$ since they're both proportional to $\frac{1}{A}$, but the goal is not to increase the resistance. We want to increase the **change** in resistance for a change in force. Whether you take the integral, nothing, or the derivative, they all increase from decreasing A, but you only get points for correct justification if you demonstrated this specifically for the sensitivity. It is also acceptable to demonstrate that the coefficient in front of F controls how much R changes for a unit of force, and you basically want to increase that (same as saying increase the slope).
- Not mathematically justifying answers, or justifications that don't quite fit the problem. Here are some examples:
  - "Decreasing A will increase sensitivity since it's in the denominator" but with no mathematical justification. You have to show that A is in the denominator of the *sensitivity* before making this claim.
  - "R going up implies S goes up" but not saying why.
  - "$R = pL/A$ so therefore small A results in big R." Unfortunately, this wasn't the question. We're interested in the effect of A on sensitivity, not total resistance.
- Trying to solve this like a maximization problems. In optimization problems, you often go straight to setting the derivative to 0 to find critical points, but these are for optimizing the parent function, in this case $R(F)$. We want to maximize sensitivity. This means you actually need $\frac{dR}{dF}$ to be large, not 0.

9. Rain Sensor (20 points)

A capacitive sensor can be used to measure how much rain water is in a tank. To do this, two capacitor plates are attached to a rectangular tank. The idea is to make a capacitor whose capacitance varies with the amount of water inside. The width and length of the tank are both w (i.e. the base is square), and the height of the tank is h_{tot}. The permittivity of air is ε, and the permittivity of rainwater is 81ε.



(a) (5 points) Derive an expression for the total capacitance, C_{tank}, between terminals a and b as a function of h_{tot}, w, $h_{\text{H}_2\text{O}}$, and ε.

Solution: The total capacitance C_{tank} contains two components: the capacitance due to water, and the capacitance due to air. Because these capacitors are parallel to each other, the total capacitance is the sum of C_{air} and $C_{\text{H}_2\text{O}}$.

The capacitance of the two plates due to water is:

$$C_{\text{H}_2\text{O}} = \frac{81\varepsilon h_{\text{H}_2\text{O}} w}{w} = 81\varepsilon h_{\text{H}_2\text{O}}$$

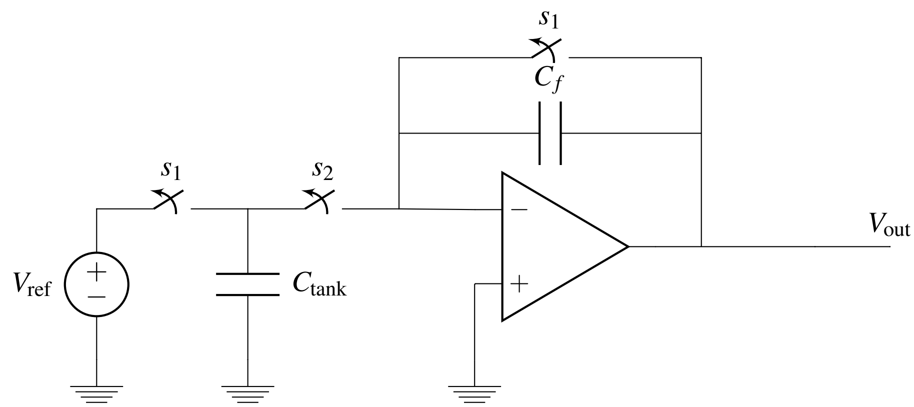
The capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\varepsilon\left(h_{\text{tot}} - h_{\text{H}_2\text{O}}\right) w}{w} = \varepsilon\left(h_{\text{tot}} - h_{\text{H}_2\text{O}}\right)$$

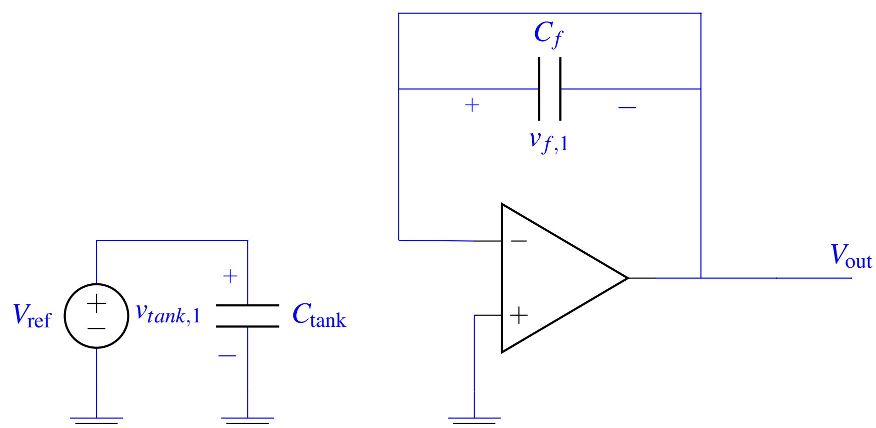
The equivalent total capacitance is then:

$$C_{\text{tank}} = C_{\text{H}_2\text{O}} + C_{\text{air}} = \varepsilon(h_{\text{tot}} + 80h_{\text{H}_2\text{O}})$$

(b) (2 points) You would like to measure changes in C_{tank} as changes in voltage, so you design the following circuit. Draw the equivalent circuit when s_1 is on and s_2 is off.



Solution:



(c) (3 points) Find expressions for charge across each capacitor when s_1 is on and s_2 is off.

Solution: The charge across C_{tank} is:

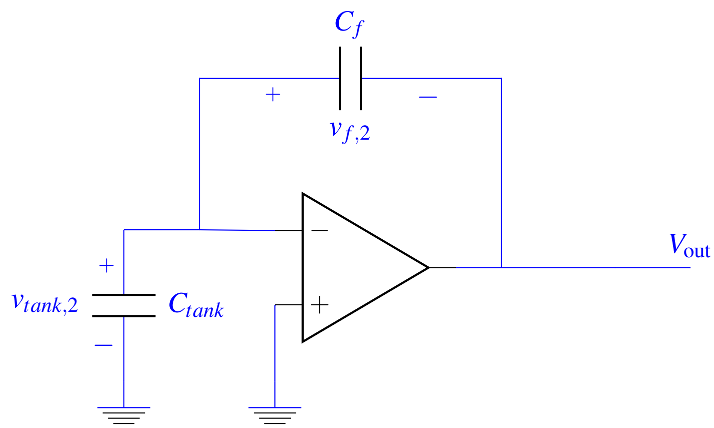
$$Q_{ctank,1} = v_{tank,1} C_{\text{tank}} = V_{\text{ref}} C_{\text{tank}}$$

The two terminals of C_f are shorted, so there is no charge across it:

$$Q_{Cf,1} = C_f V_{f,1} = 0$$

(d) (2 points) Draw the equivalent circuit when s_2 is on and s_1 is off.

Solution:



(e) (3 points) Find expressions for charge across each capacitor when s_2 is on and s_1 is off.

Solution: Since the voltage on the positive terminal of this op amp in negative feedback is zero, the voltage on the negative terminal must also be zero. The charge across C_{tank} is given by:

$$Q_{ctank,2} = C_{\text{tank}} v_{tank,2} = 0$$

The charge across C_f is given by

$$Q_{f,2} = C_f v_{f,2} = C_f(0 - V_{out}) = -C_f V_{out}$$

(f) (5 points) Express V_{out} as a function of V_{ref} and the capacitances. What happens to V_{out} when the amount of water in the tank increases?

Solution: Using the fact that the charge on a node is conserved, we can write the following expression

$$Q_{ctank,1} + Q_{Cf,1} = Q_{ctank,2} + Q_{Cf,2}$$

After plugging in the expressions in the previous parts:

$$V_{\text{ref}} C_{\text{tank}} = -C_f V_{out}$$

$$V_{out} = -\frac{C_{\text{tank}}}{C_f} V_{\text{ref}}$$

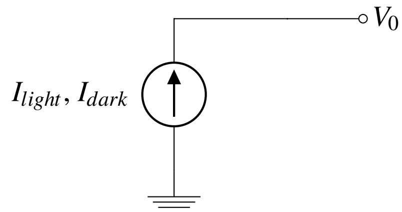
As the amount of water increases, the capacitance C_{tank} increases, causing the voltage at the output to decrease.

PRINT your name and student ID: ______________________________

10. A Light Design Problem, So to Speak... (15 points)

Your pet cactus needs a lot of light every day, and you're not sure your room is sunny enough. You decide to use your knowledge from 16A to build a device to detect if the sun is shining in your room.

(a) (8 points) Assume you just finished the Imaging Lab, where you build a light sensor, which can be modeled as a current source as shown below. When the light is on, the current source produces 5 mA. When the light is off, 1 mA.



$I_{light} = 5$ mA, $I_{dark} = 1$ mA

Using this knowledge, **design a circuit that outputs 10 V when the light is on and 0 V when the light is off**. In addition to the model of the light sensor above, you may use the following components (only!):

- **one** op amp
- **one** resistor
- **two** voltage sources

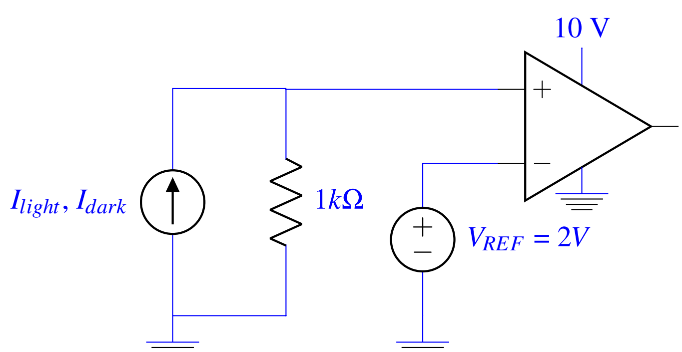
You must explicitly note the power supplies used on the op amp. Clearly label values of resistors and voltage sources that you use.

Solution:

We want to utilize the fact that the light being on and off will produce different voltages in our sensor. Picking a resistor value of $R = 1k\Omega$, we can see the voltages for light being on and off would be:

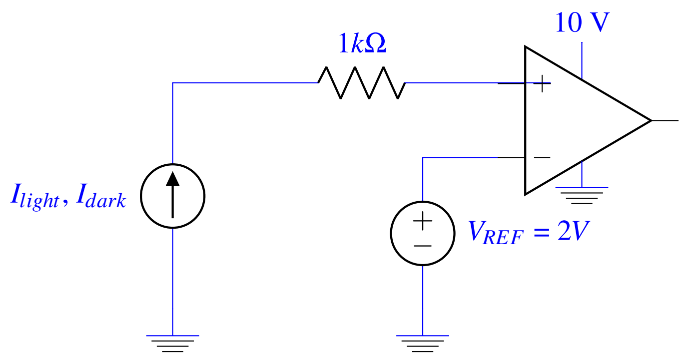
$$V_{light} = (1k\Omega)(5mA) = 5V$$
$$V_{dark} = (1k\Omega)(1mA) = 1V$$

Now that we have two separate voltage values, we can run our sensor into a comparator, whose V_{REF} value is between V_{light} and V_{dark}, as shown. Note that the op amp's power supply rails have been chosen to be $V_{DD} = 10V$ and $V_{SS} = 0V$ to match our desired output.



Common Mistakes:

The following is circuit design is incorrect because the current source is feeding directly into the op amp, but no current can enter the op amp. Therefore, this circuit will not output the desired voltages.



(b) (4 points) You realize that the light sensor can only provide a maximum of $P_{max} = 40mW$ of power. Does this affect your design from part (a)? If so, how should it change? If not, why does it not affect it?

(Do not use additional components beyond those specified in the problem statement.)

Solution: The max power provided by the sensor will happen when the light is on, $I = 5mA$. In the design above, this creates a voltage drop over the sensor of $-IR = -(5mA)(1k\Omega) = -5V$ (using passive sign convention). The power dissipated by the sensor is

$$P = IV = (5mA)(-5V) = -25mW$$

The power *dissipated* negative, so the power *provided* to the circuit is positive $25mW$ which is less than P_{max}. Therefore this circuit does not need to be altered.

Let's calculate what resistance values meet the maximum power requirement:

$$\begin{aligned} |P| &\leq 40mW \\ |IV| &\leq 40mW \\ |I^2V| &\leq 40mW \\ |(5mA)^2R| &\leq 40mW \\ R &\leq 1.6k\Omega \end{aligned}$$

Common Mistakes:

- A common mistake was to calculate the maximum possible voltage $P_{max} = IV_{max}$ using 40 mW and 5 mA or 1mA and getting $V_{max} = 8V, 40V$, then use these voltages as the output of the light sensor. However, these are just the maximum possible voltages and not the actual voltages. The actual voltages and power bounds of the light sensor depend on the resistor value. That is, you have to show that your light sensor's voltage is less than these maximum voltages.
- The power constraint only applies to the light sensor (the current source). Many people mistakenly thought the power constraint applied to the op amp.

(c) (3 points) You notice that there is noise in both I_{light} and I_{dark} (ie. sometimes the currents are higher or lower than expected). Assuming that there is the same amount of noise in both I_{light} and I_{dark}, modify your design to be as robust as possible to these fluctuations. Justify your design choices in 1-2 sentences. If you believe your design does not need modification, explain why.

(Do not use additional components beyond those specified in the problem statement.)

Solution: The reference voltage should be exactly in between V_{light} and V_{dark} to be most robust to noise in the inputs. Specifically,

$$V_{ref} = \frac{I_{light} + I_{dark}}{2} R$$

. Using the values for I_{light} and I_{dark}, the expression then becomes:

$$V_{ref} = \frac{6mA}{2} R = \frac{6mA}{2} 1k\omega = 3V$$

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