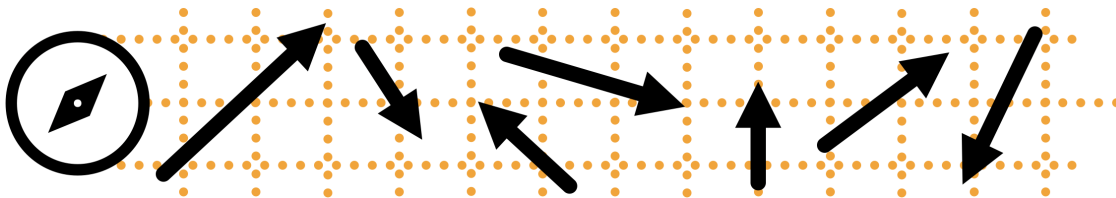


EECS 16A Designing Information Devices and Systems I

Fall 2021 Discussion 2B

1. Vectors



A vector is an ordered list of numbers. For instance, a point on a plane (x,y) is a vector! We label vectors using an arrow overhead \vec{v} , and since vectors can live in ANY dimension of space we'll need to leave our notation general $(x,y) \rightarrow \vec{v} = (v_1, v_2, \dots)$. Below are few more examples (the left-most form is the general definition):

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \qquad \vec{b} = \begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix} \in \mathbb{R}^2$$

Just to unpack this a bit more, $\vec{b} \in \mathbb{R}^3$ in english means "vector \vec{b} lives in 3-Dimensional space".

- The \in symbol literally means "in"
- The \mathbb{R} stands for "real numbers" (FUN FACT: \mathbb{Z} means "integers" like $-2, 4, 0, \dots$)
- The exponent \mathbb{R}^n indicates the dimension of space, or the amount of numbers in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{x}$ above. We call these *column vectors*, in contrast to horizontally written vectors which we call *row vectors*.

Okay, let's dig into a few examples:

(a) Which of the following vectors live in \mathbb{R}^2 space?

$$i. \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad ii. \begin{bmatrix} 5 \\ 0 \\ 3 \\ 5 \end{bmatrix} \qquad iii. \begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix} \qquad iv. \begin{bmatrix} -20 \\ 100 \end{bmatrix}$$

Answer:

i. Yes ii. No iii. No iv. Yes

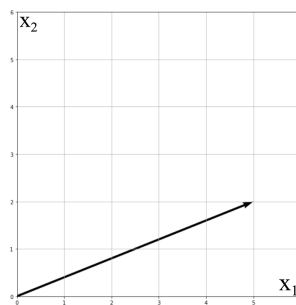
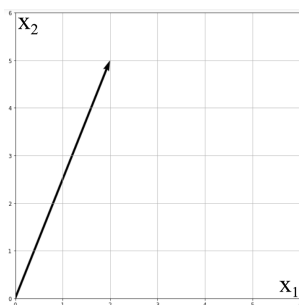
Remember \mathbb{R}^2 means 2D space, which hosts vectors with 2 terms.

We count and see only i. and iv. have 2 terms.

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):

$$i. \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad ii. \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

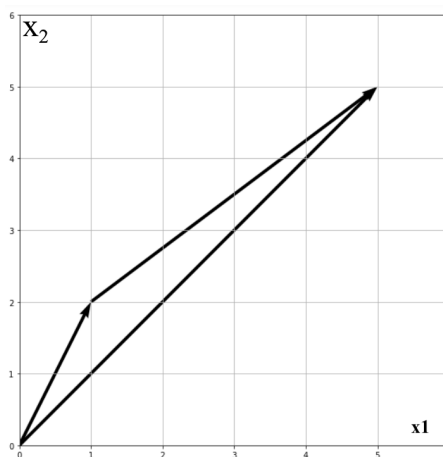
Answer: Although these vectors look similar, remember that the ordering matters!



- (c) Compute the sum $\vec{a} + \vec{b} = \vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also is there only one possible triangle?)

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Answer:



Computation is done element-wise:

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

2. Computations: Inner product and matrix-vector multiplication

- (a) For each of the following pairs of vectors, compute their inner product.

i.

$$\vec{a} = \begin{bmatrix} 1 \\ 6 \\ 11 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -6 \\ 1 \\ 2 \end{bmatrix}$$

ii.

$$\vec{a} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ -4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Answer:

i. $\vec{a}^T \vec{b} = (1)(-6) + (6)(1) + (11)(2) = 22.$

$$\text{ii. } \vec{a}^T \vec{b} = (2)(6) + (6)(2) + (12)(-1) + (-4)(3) = 0.$$

(b) Perform matrix vector multiplication to compute $A\vec{b}$ in each of the following cases:

i.

$$A = \begin{bmatrix} 1 & 6 \\ 2 & -7 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

ii.

$$A = \begin{bmatrix} 1 & 9 & 2 \\ 7 & 10 & -7 \\ -1 & 2 & -8 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Answer:

i. Let $a_i \in \mathbb{R}^3$ represent the transpose of the i_{th} row of the A matrix. To find the i_{th} entry of the $A\vec{b}$ vector we find the inner product of a_i and \vec{b} . In this case, we get

$$A\vec{b} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

ii. Let $A_i \in \mathbb{R}^3$ represent the i_{th} column of A and let $b_i \in \mathbb{R}$ represent the i_{th} component of \vec{b} .

$$A\vec{b} = A_1 b_1 + A_2 b_2 + A_3 b_3 = 1 \times \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} + 3 \times \begin{bmatrix} 2 \\ -7 \\ -8 \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \\ -25 \end{bmatrix}$$

3. Mini-Lecture: Matrix-Vector Form for Systems of Linear Equations

Consider a system of linear equations with 3 unknowns x_1, x_2, x_3 . As we have seen, such a system of equations can be written in the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned} \tag{1}$$

where the a_{ij} and b_i are all real-valued constants.

We previously introduced the augmented matrix notation, which was useful for systematically determining solutions (e.g., using Gaussian elimination). Now, using matrix-vector multiplication, observe that we can also write this system in **matrix-vector form**:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2}$$

If we denote the coefficient matrix by A , the variable vector by \vec{x} and the vector of constant terms by \vec{b} then the system of equations can be concisely written as

$$A\vec{x} = \vec{b}.$$

This is called the **matrix-vector form** (or **matrix-vector representation**) of the system of linear equations Eq. (1).

Now that we understand how to multiply matrices with vectors, we can revisit this representation of systems of equations, to demonstrate that it is algebraically equivalent to the original system.

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{13}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3}$$

Notice that the left-hand-sides of our original system of equations have suddenly appeared as components of the vector on the left-hand-side! Since equating two vectors is the same as equating their corresponding coefficients, we can rewrite the above vector equation as a set of n scalar equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned} \tag{4}$$

which are exactly the equations in our linear system.

4. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, *if the product exists*, find the product by hand. Otherwise, explain why the product does not exist.

- (a) **AB Answer:** A is a 1×2 vector and B_1 is a 2×1 vector, so the product exists!
 $\mathbf{AB} = 1 \cdot 3 + 4 \cdot 2 = 11$

- (b) **CD Answer:** Since both C and D are 2×2 matrices, the product exists and is a 2×2 matrix.
 $\mathbf{CD} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 4 \cdot 2 & 1 \cdot 2 + 4 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 2 & 2 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}$.

- (c) **DC Answer:** Since both C and D are 2×2 matrices, the product exists and is a 2×2 matrix.
 $\mathbf{DC} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 2 \cdot 2 & 3 \cdot 4 + 2 \cdot 3 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 4 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}$.

- (d) **CE Answer:** Since C is a 2×2 matrix and E is a 2×4 matrix, the product exists and is a 2×4 matrix.
 $\mathbf{CE} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 4 & 1 \cdot 9 + 4 \cdot 3 & 1 \cdot 5 + 4 \cdot 2 & 1 \cdot 7 + 4 \cdot 2 \\ 2 \cdot 1 + 3 \cdot 4 & 2 \cdot 9 + 3 \cdot 3 & 2 \cdot 5 + 3 \cdot 2 & 2 \cdot 7 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}$.

- (e) **FE** (only note whether or not the product exists) **Answer:**
 Since E is a 2×4 matrix and F is a 4×3 matrix, the product does not exist.
 This is because the number of columns in the first matrix (F) should match the number of rows in the second matrix (E) for this product to be defined.

- (f) **EF** (only note whether or not the product exists) **Answer:**
 Since E is a 2×4 matrix and F is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\mathbf{EF} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 5 + 9 \cdot 6 + 5 \cdot 4 + 7 \cdot 3 & 1 \cdot 5 + 9 \cdot 1 + 5 \cdot 1 + 7 \cdot 2 & 1 \cdot 8 + 9 \cdot 2 + 5 \cdot 7 + 7 \cdot 2 \\ 4 \cdot 5 + 3 \cdot 6 + 2 \cdot 4 + 2 \cdot 3 & 4 \cdot 5 + 3 \cdot 1 + 2 \cdot 1 + 2 \cdot 2 & 4 \cdot 8 + 3 \cdot 2 + 2 \cdot 7 + 2 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}$$

(g) **GH Answer:** Since **G** and **H** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8 \cdot 5 + 1 \cdot 1 + 6 \cdot 2 & 8 \cdot 3 + 1 \cdot 8 + 6 \cdot 3 & 8 \cdot 4 + 1 \cdot 2 + 6 \cdot 5 \\ 3 \cdot 5 + 5 \cdot 1 + 7 \cdot 2 & 3 \cdot 3 + 5 \cdot 8 + 7 \cdot 3 & 3 \cdot 4 + 5 \cdot 2 + 7 \cdot 5 \\ 4 \cdot 5 + 9 \cdot 1 + 2 \cdot 2 & 4 \cdot 3 + 9 \cdot 8 + 2 \cdot 3 & 4 \cdot 4 + 9 \cdot 2 + 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **HG Answer:** Since **H** and **G** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{HG} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 8 + 3 \cdot 3 + 4 \cdot 4 & 5 \cdot 1 + 3 \cdot 5 + 4 \cdot 9 & 5 \cdot 6 + 3 \cdot 7 + 4 \cdot 2 \\ 1 \cdot 8 + 8 \cdot 3 + 2 \cdot 4 & 1 \cdot 1 + 8 \cdot 5 + 2 \cdot 9 & 1 \cdot 6 + 8 \cdot 7 + 2 \cdot 2 \\ 2 \cdot 8 + 3 \cdot 3 + 5 \cdot 4 & 2 \cdot 1 + 3 \cdot 5 + 5 \cdot 9 & 2 \cdot 6 + 3 \cdot 7 + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$