

EECS 16A Designing Information Devices and Systems I

Discussion 4A

1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of \mathbf{A} exists.
If it exists, compute the inverse using Gauss-Jordan method.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer: We use Gaussian elimination (also known as the Gauss-Jordan method):

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftarrow \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array} \right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Answer: We can again use the Gauss-Jordan method:

$$\begin{aligned} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] &\xrightarrow{R_1 \leftarrow \frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \leftarrow R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{c}{a}b & -\frac{c}{a} & 1 \end{array} \right] \\ &\xrightarrow{R_2 \leftarrow \frac{1}{d - \frac{c}{a}b}R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d - \frac{c}{a}b} & \frac{1}{d - \frac{c}{a}b} \end{array} \right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right] \\ &\xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{b}{a} \frac{-c}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right]. \end{aligned}$$

Therefore, we get that $\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This is a known formula which, if you find useful, you can use for any general 2x2 matrix.
Note that the matrix does not have an inverse if $ad - bc = 0$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$

Answer:

Since have a non-square matrix \mathbf{A} , there cannot be a unique inverse.

We can understand this from the fact that for $\vec{y} = \mathbf{A}\vec{x}$ the vectors $\vec{x} \in \mathbb{R}^3$ and $\vec{y} \in \mathbb{R}^2$ live in different spaces. This leads us to conclude that there cannot be a unique \vec{x} for each \vec{y} .

$$(d) \mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$$

Answer: We use Gaussian elimination:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array} \right]. \end{aligned}$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

2. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \mid \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of \mathbb{R}^3 ? Why/why not?

Answer:

Yes, V is a subspace of \mathbb{R}^3 . We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of \mathbb{R}^3 – all elements in V are of the form $\begin{bmatrix} c+d \\ c \\ c+d \end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$,

such that $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (\alpha d_1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\alpha \vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V , when we set $c = 0$ and $d = 0$.

We have thus identified V as a subset of \mathbb{R}^3 , shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so V is a subspace of \mathbb{R}^3 .

It's important to note that satisfying the subset property and the two forms of closure additionally implies this subspace V also satisfies the axioms of a vector space, and therefore is definitionally also a vector space.

3. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

- What is the column space of \mathbf{A} ? What is its dimension?
- What is the null space of \mathbf{A} ? What is its dimension?
- Are the column spaces of the row reduced matrix \mathbf{A} and the original matrix \mathbf{A} the same?
- Do the columns of \mathbf{A} span \mathbb{R}^2 ? Do they form a basis for \mathbb{R}^2 ? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer: Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

The column space does not span \mathbb{R}^2 and thus are not a basis for \mathbb{R}^2 .

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same.

Not a basis for \mathbb{R}^2 .

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: \mathbb{R}^2

Null space: $\text{span} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

The two column spaces are the same as the column span \mathbb{R}^2 .

This is a basis for \mathbb{R}^2 .

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: $\text{span} \left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: $\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

The two column spaces are not the same.

Not a basis for \mathbb{R}^2 .

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

- i. The column space of the columns is \mathbb{R}^2 . The columns of \mathbf{A} do not form a basis for \mathbb{R}^2 . This is because the columns of \mathbf{A} are linearly dependent.
- ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on \mathbf{A} . We start by performing Gaussian elimination on matrix \mathbf{A} to get the matrix into upper-triangular form.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{ reduced row echelon form} \end{aligned}$$

$$x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0$$

$$x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0$$

x_3 is free and x_4 is free

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$x_1 + \frac{1}{2}s - \frac{7}{2}t = 0$$

$$x_2 + \frac{5}{2}s + \frac{1}{2}t = 0$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$x_1 = -\frac{1}{2}s + \frac{7}{2}t$$

$$x_2 = -\frac{5}{2}s - \frac{1}{2}t$$

$$y = s$$

$$z = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the nullspace of \mathbf{A} can be written as follows:

$$\text{Nullspace}(\mathbf{A}) = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of \mathbf{A} is

$$\text{span} \left\{ \left[\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right] \right\}$$

\mathbf{A} has a 2-dimensional null space.

- iii. In this case, the column space of the row reduced matrix is also \mathbb{R}^2 , but this need not be true in general.
- iv. No, the columns of \mathbf{A} do not form a basis for \mathbb{R}^2 .

4. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space \mathbb{R}^k and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in \mathbb{R}^k .

- (a) For the first part of the problem, let $k > n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^k ? Why/why not? What conditions would we need?

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ cannot form a basis for \mathbb{R}^k . The dimension of \mathbb{R}^k is k , so you would need k linearly independent vectors to describe the vector space. Since $n < k$, this is not possible.

- (b) Let $k = n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^k ? Why/why not? What conditions would we need?

Answer:

Fact: matrix V is invertible $\iff V$ is square and has linearly independent columns

Note you have not proven this, but you will see the proof in EECS16B. Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are k of them, we can put them into a square matrix V :

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

This matrix is square because the number of entries in the column vectors (k) is equal to the number of column vectors (n).

Using the fact from above, we know that if the square matrix V has n linearly independent columns, it will be invertible. If the matrix V is invertible, the matrix vector equation $V\vec{x} = \vec{b}$ will always have a unique solution for all vectors \vec{b} . Thus all possible $\vec{b} \in \mathbb{R}^k$ are in the span of the columns of matrix V : $(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\})$.

To summarize, we can conclude then that if $n = k$ and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for \mathbb{R}^k .

- (c) Now, let $k < n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^k ? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ cannot form a basis for \mathbb{R}^k . \mathbb{R}^k will be spanned by k linearly independent vectors. Any additional vectors in \mathbb{R}^k must already exist in the span of the previous vectors, and are therefore linearly dependent. Since $n > k$, some of the vectors have to be linearly dependent, so they cannot form a basis.

The two regimes—one where $n > k$ and one where $n < k$ —give rise to two different classes of interesting problems. You might learn more about them in upper division courses!