# EECS 16A Dis 3B Notes

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**1. Inverses**

In general, the *inverse* of a matrix "undoes" the operation that a matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

$5 \cdot \frac{1}{5} = 1$

$x \cdot x^{-1} = 1$

$A \cdot A^{-1} = I$

**Properties of Inverses**

For a matrix $\mathbf{A}$, if its inverse exists, then:

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

$(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$ for a nonzero scalar $k \in \mathbb{R}$

$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ $T$ is "Transpose"

$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ assuming $\mathbf{A}, \mathbf{B}$ are both invertible

$\underbrace{AB\cancel{C} \cdot B^{-1}}_{I} C^{-1}A^{-1} \neq I$

$B^{-1}C^{-1}A^{-1} \neq (ABC)^{-1}$

(a) Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all invertible matrices. Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

$A \cdot B \cdot C \cdot (ABC)^{-1} = I$

$(ABC)C^{-1}$

$A \cdot B \cdot \underbrace{C \cdot C^{-1}}_{I} = A \cdot B \cdot B^{-1} = A \cdot A^{-1} = I$

$\boxed{C^{-1} \cdot B^{-1} \cdot A^{-1}}$

$c_i = 0 \rightarrow LI$

(b) Now consider the following four matrices.

$c_1\vec{a}_1 + c_2 a_2 + \dots + c_n\vec{a}_n = 0$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\cancel{c_i = 0}$

$c_i \neq 0$

L.D vectors

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.

A: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ B: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

$d_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

C: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{bmatrix}$ D: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$

$d_1 = 0$ $d_2 = 100 = 0$

ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.

No

$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?

No → A, B, C, D don't have LI cols/rows

iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$? (*Bonus:* How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)

Linearly dependent

If $b \in col(A)$ & A has lindep cols!

$\infty$ sols

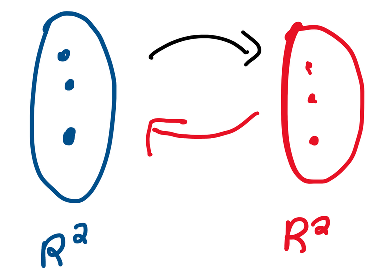
v. Are all square matrices invertible?

No

L.I - linearly independent

L.D - linearly dependent

**Invertibility:**




$A\vec{x} = \vec{b}$

$\vec{x} = A^{-1}\vec{b}$

Invertibility:

- A is a square matrix
- A's cols/rows are LI

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

**2. Visualizing Matrices as Operations**

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

**Part 1: Rotation Matrices as Rotations**

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30°, respectively. Suggest some methods to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by 60°? Your TA will show you the result in the iPython notebook.

$T_1 x \rightarrow$ rotates x by 15°

$T_2 x \rightarrow$ rotates x by 30°

$T_2(T_1 x) \rightarrow$ rotates x by 45°

45° rotation

(b) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by 60°. Your TA will show you the result in the iPython notebook.

$2 \cdot T_2$ $4 \cdot T_1$

$T_2 \cdot T_2$ $T_1^4$

$T_2^2 \rightarrow \underbrace{T_2 \cdot \underbrace{T_2 \cdot x}_{30°}}_{30°}$

$T_1^4 \cdot x \rightarrow 60°$ rotation

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle $\theta$. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta$ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ by $\theta$ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \dots} = 1$.)

(*Hint: Use your trigonometric identities!*)

1. Trignometric identities

$\begin{bmatrix} \cos(\theta + \alpha) \\ \sin(\theta + \alpha) \end{bmatrix}$

$$R \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos(\theta + \alpha) \\ \sin(\theta + \alpha) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \sin\alpha\cos\theta \end{bmatrix}}$$

$\Leftrightarrow$ $\boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ R

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

Rotation

$R \cdot x$: CCW by $\theta$

$R^{-1} \cdot (Rx)$: CCW by $-\theta$

Rotate by $\theta$

Undo by $-\theta$

Rotation: CCW by $\theta$

Inverse: CCW by $-\theta$

(e) Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by $\theta$. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \rightarrow \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \leftarrow \boxed{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$$

Inverse

(f) What are the matrices that reflect a vector about the (i) $x$-axis, (ii) $y$-axis, and (iii) $x = y$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$

**Part 2: Commutativity of Operations**

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Your

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

the $y$-axis.

$T_4 \cdot T_3 \neq T_3 T_4$

reflection rotation

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

Reflections are not commutative in general

**Part 3: Distributivity of Operations**

(a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.

$A(v_1 + v_2)$

$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}\left(\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} + \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}\right)$

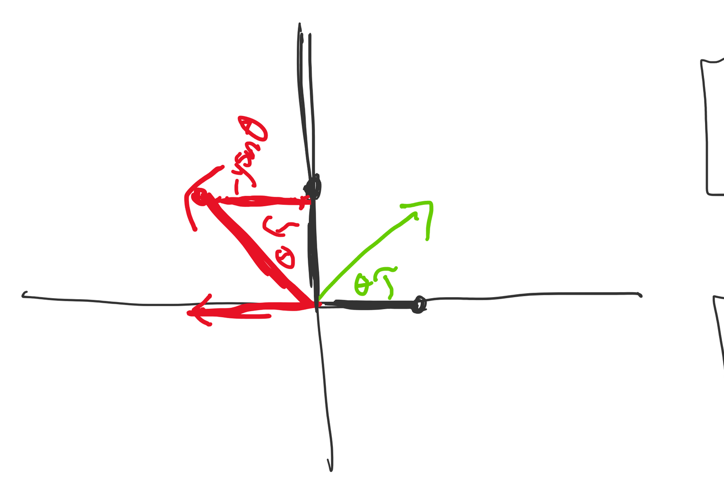
$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}\begin{bmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \end{bmatrix} = \vec{a}_1(v_{11} + v_{21}) + \vec{a}_2(v_{12} + v_{22})$

$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \end{bmatrix} = \begin{bmatrix} a_{11}(v_{11} + v_{21}) + a_{21}(v_{12} + v_{22}) \\ a_{12}(v_{11} + v_{21}) + a_{22}(v_{12} + v_{22}) \end{bmatrix}$

$= Av_1 + Av_2$

2. Linearity

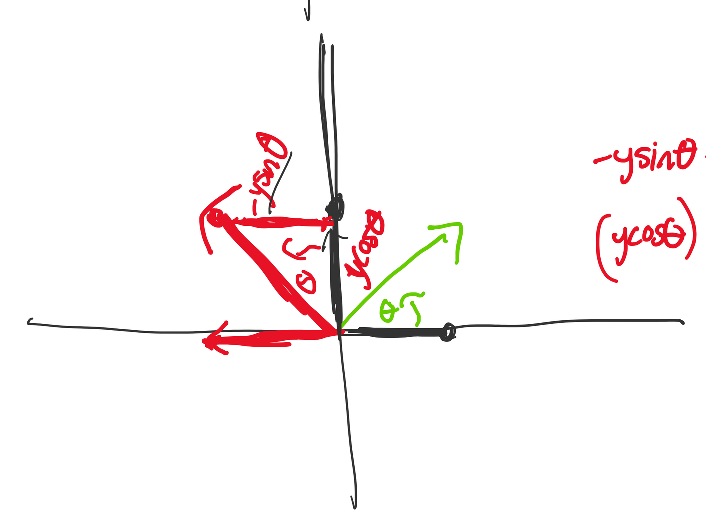
$R\begin{bmatrix} x \\ y \end{bmatrix} = R\begin{bmatrix} x \\ 0 \end{bmatrix} + R\begin{bmatrix} 0 \\ y \end{bmatrix}$



$\begin{bmatrix} x \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x\cos\theta \\ x\sin\theta \end{bmatrix}$

$\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -y\sin\theta \\ y\cos\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$



$-y\sin\theta$ - x-component

$(y\cos\theta)$ y-component

$\cos(\theta)$ $x^2$

Even: $f(-x) \rightarrow f(x)$

$f(-x) = f(x)$

$f(-x) \rightarrow -f(x)$

$f(x) \rightarrow$

Odd: $x \rightarrow f(x)$

$-x \rightarrow -f(x)$

$\sin(\theta)$, $x$, $x^3$, $x^5$