EECS 16A Dis 4A Notes

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1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of **A** exists.
If it exists, compute the inverse using Gauss-Jordan method.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$A\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 9 & 0 \end{array}\right]$ $\vec{x} = \vec{0}$

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$

$A\vec{x} = \vec{b}$

$A^{-1} \cdot A\vec{x} = A^{-1}\vec{b}$

$I \cdot \vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1} \cdot \vec{b}$

(b) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Invertibility:
1. A is a square matrix
2. A has LI cols

$\frac{1}{a}\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$R_1 = \frac{1}{a}R_1$ $\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$

$R_2 = -cR_1 + R_2$ $\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{-bc}{a} + d & \frac{-c}{a} & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{-bc+da}{a} & \frac{-c}{a} & 1 \end{array}\right]$

$A\vec{x} = \vec{b}$ $[A \mid b]$

$A \cdot A^{-1} = I$ $[A \mid I]$

$R_2 = \frac{a}{-bc+da}R_2$ $\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{-bc+da} & \frac{a}{-bc+da} \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{-bc+da} & \frac{-b}{-bc+da} \\ 0 & 1 & \frac{-c}{-bc+da} & \frac{a}{-bc+da} \end{array}\right]$

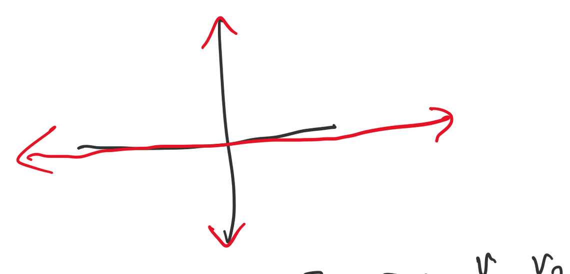
$\frac{1}{-bc+da}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$

No inverse

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

No inverse



Subspace: "subset of a vector space"
1) Existence of $\vec{0}$
2) Closed under vector addition
3) Closed under scalar multiplication

$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ $v_1 \; v_2 \; v_3$

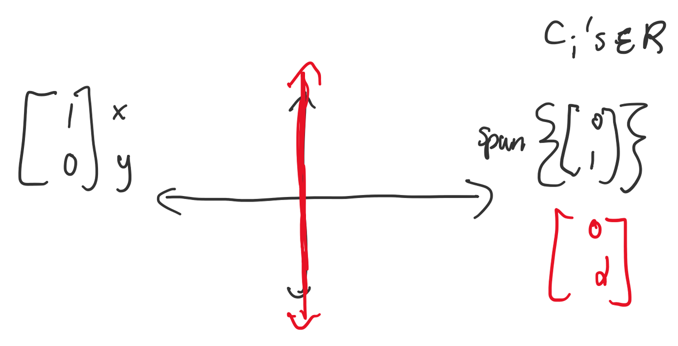
$c_1v_1 + c_2v_2 + c_3v_3$

$c_i\text{'s} \in R$

Column space: Span{column vectors of A}

$Col(A) = C(A)$

$A\vec{x} = \vec{b}$



Rank: # of LI vectors in matrix A
# of pivot columns in RREF

Nullspace: Solutions to $A\vec{x} = \vec{0}$

$\{\vec{x} \mid A\vec{x} = \vec{0}\}$

Basis: Basic set of vectors needed to span a vector space V
1. $\{b_1 \ldots b_n\}$ are LI
2. $\text{span}\{b_1 \ldots b_n\} = R^n$

$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = R^2$

≠ basis

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

2. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

1. $\vec{0} \in$ subspace

$c = 0 \quad d = 0 \quad 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$ ✓

2. $v_1 = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$v_2 = c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ✓ (scalar, scalar)

$\alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. $v_1 = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\alpha v_1 = \alpha c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha d_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ✓ (scalar, scalar)

Subspace :)

3. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension? → # of basis vectors
ii. What is the null space of **A**? What is its dimension?
iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?
iv. Do the columns of **A** span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

i. $col(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ Dim = 1

ii. $N(A)$ $A\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$N(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$\begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = \alpha$, $0 = 0$

$\begin{bmatrix} 0 \\ \alpha \end{bmatrix}$

$\text{Span}\left\{\begin{bmatrix} 0 \\ \alpha \end{bmatrix}\right\}$

$\text{Span}\left\{\alpha\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

iii. $Col(RREF(A)) = Col(A)$ Same

$RREF(A) = A$

iv. ~~Not~~ Doesn't span R^2; not a basis for R^2

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

i. $Col(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ Dim = 2

ii. $N(A)$ $A\vec{x} = \vec{0}$ $N(A) = \vec{0}$ Dim = 0

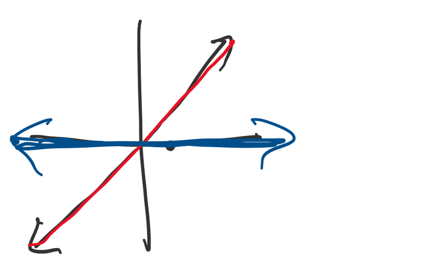
iii. $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$R_2 = \frac{1}{3}R_2$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ (2) $RREF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (2)

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $RREF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

4. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^k$ and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^k$.

(a) For the first part of the problem, let $k > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^k$? Why/why not? What conditions would we need?

$k > n$

No basis

$\text{span}\{v_1 \ldots v_n\} = R^n$

(b) Let $k = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^k$? Why/why not? What conditions would we need?

$k = n$ & $v_1 \ldots v_n$

$v_1 \ldots v_n$ LI $\Longleftrightarrow$ basis for R^k

(c) Now, let $k < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^k$? What vector space could they form a basis for?
Hint: Think about whether the vectors can be linearly independent.

$v_1 \ldots v_n \Longleftrightarrow$ LD

no basis

R^n

$R^k \Longleftrightarrow k$ vectors ~~basis~~ in basis

$v_1 \ldots v_n$

$n-k$ vectors are linearly dependent $\Longleftrightarrow$

k vectors $\Longleftrightarrow$ basis

$n > k$

k ~~vectors LI~~ LI vectors out of n vectors $\Longleftrightarrow R^k$

$n-k$ vectors L.D.

$k + (n-k) = n$

1. $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = R^2$

≠ basis

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = R^3$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Questions

1. What are quick tips to determine if columns are LI? $A\vec{x} = 0$
   - Check by inspection if cols are L.D. (are the cols scalar multiples of one another?)
   - Could also calculate the determinant of a matrix to see if it is non-zero
   - Solve $A\vec{x} = \vec{0}$ to see if $N(A)$ is non-trivial (contains more than $\vec{0}$)

2. What is a basis for R^2?

   any 2 LI vectors $\in R^2$

   $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$

   Basis conditions
   1. vectors are LI (maximum condition: there can at most be n LI vectors $\in R^n$)
   2. vectors span R^n (minimum condition: at least n vectors are required to span R^n)

   Thus, if given k vectors and determining whether those k vectors form a basis for R^n
   1. $k \leq n$ in order for the vectors to be LI
   2. $k \geq n$ in order for the vectors to span R^n

   ∴ $k = n$ (there are n LI vectors $\in R^n$ in basis for R^n)

3. Do we include the trivial case of nullspace ($\vec{x} = 0$) when we say what the dimension of the nullspace is?

   If $N(A) = \{\vec{0}\}$, then $Dim(N(A)) = 0$

   $\vec{0}$ is the trivial solution

4. Why is RREF(A) in 3c(iii) the identity matrix?

   $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   Columns of A are LI so we expect two pivot columns in RREF(A)