EECS 16A Dis 5B Notes

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1. Steady and Unsteady States

(a) You're given the matrix **M**:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$
ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$
iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Eigenvalue/eigenvector recap:

1. Set up $\boxed{Mx = \lambda x}$
2. Find what values for λ result in a non-trivial nullspace $B = M - \lambda I$ $(M - \lambda I)\vec{x} = \vec{0}$
Non-trivial nullspace $\iff$ $\text{Det}(M - \lambda I) = 0$
$\therefore$ have to solve char. polynomial for λ_i's
3. Plug in λ_i's to solve for the vectors in the non-trivial nullspace $\rightarrow$ eigenvectors

Distinct e-vals $\rightarrow$ e-vecs LI

i. $\lambda_1 = 1$ $(M - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1-1 & -2 \\ 0 & 0 & 2-1 \end{bmatrix} \vec{x} = \vec{0} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \vec{0} \qquad \boxed{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \qquad \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

ii. $\lambda_2 = 2$

$$\begin{bmatrix} \frac{1}{2}-2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1-2 & -2 \\ 0 & 0 & 2-2 \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0} \qquad v_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

iii. $\lambda_3 = \frac{1}{2}$

$$\begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \vec{x} = \vec{0} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$x_1 = x_1$

eigenvectors form a basis for $\mathbb{R}^3$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x} \qquad x[n] = ?$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty}\mathbf{M}^n\vec{x}$
0	0	$\neq 0$	Yes	0
0	$\neq 0$	0	No	/
0	$\neq 0$	$\neq 0$	No	/
$\neq 0$	0	0	Yes	αv_1
$\neq 0$	0	$\neq 0$	Yes	αv_1
$\neq 0$	$\neq 0$	0	No	/
$\neq 0$	$\neq 0$	$\neq 0$	No	/

$$M^n \cdot x = \alpha(1)^n v_1 + \beta(2)^n \cdot v_2 + \gamma \cdot \left(\tfrac{1}{2}\right)^n \vec{v}_3$$

$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = \frac{1}{2}$

$2^n \rightarrow \infty$ as $n \rightarrow \infty$

$\underbrace{\gamma \cdot \left(\tfrac{1}{2}\right)^n \cdot \vec{v}_3}_{0}$ as $n \rightarrow \infty$

$x[1] = M \cdot x[0]$
$x[2] = M \cdot x[1] = M^2 x[0]$
$\vdots$
$x[n] = M^n \cdot x[0]$

$x = \alpha v_1 + \beta v_2 + \gamma v_3$
$M \cdot x = M(\alpha v_1 + \beta v_2 + \gamma v_3)$
$= M\alpha v_1 + M\beta v_2 + M\gamma v_3$
$= \alpha M v_1 + \beta M v_2 + \gamma M v_3$ $\quad Mx = \lambda x$
$= \alpha \cdot \lambda_1 v_1 + \beta \cdot \lambda_2 v_2 + \gamma \cdot \lambda_3 v_3$

$$\boxed{M^n \cdot x = \alpha \cdot \lambda_1^n v_1 + \beta \lambda_2^n \cdot v_2 + \gamma \cdot \lambda_3^n v_3}$$

$Mx = \lambda x$
$M \cdot Mx = M \cdot \lambda x$
$= \lambda(Mx)$
$= \lambda \cdot \lambda x$
$M^2 x = \lambda^2 x$
$\vdots$
$M^n x = \lambda^n x$

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

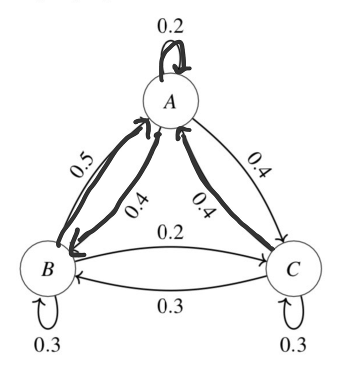

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix **T** representing the pumps system.

$x[A] = .2x[A] + .5x[B] + .4x[C]$
$x[B] = .4x[A] + .3x[C] + .3x[B]$
$x[C] = .4x[A] + .2x[B] + .3x[C]$

$$\underbrace{\begin{bmatrix} .2 & .5 & .4 \\ .4 & .3 & .3 \\ .4 & .2 & .3 \end{bmatrix}}_{T} \begin{bmatrix} x[A] \\ x[B] \\ x[C] \end{bmatrix} = \begin{bmatrix} x[A] \\ x[B] \\ x[C] \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of **T**. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

eigenvectors for $\lambda = 1$

$T \cdot \vec{x} = \vec{x}$
$T \cdot \vec{x} = \lambda\vec{x}$ $\quad T\vec{x} = \vec{x}$
$\lambda = 1$

$T - \lambda I = T - 1 \cdot I$

$$\left(\begin{bmatrix} .2 & .5 & .4 \\ .4 & .3 & .3 \\ .4 & .2 & .3 \end{bmatrix} - \lambda I \right) \vec{x} = 0 \quad \ldots \rightarrow \begin{bmatrix} 1 & 0 & -\frac{43}{36} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - \frac{43}{36}x_3 = 0$
$x_2 - \frac{10}{9}x_3 = 0$
$x_3 = x_3$

$$x_3 \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix} \qquad \alpha \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix}$$

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

$\lambda_1 = 1 \quad \lambda_2 = \left(\frac{-\sqrt{2}-1}{10}\right) < 1 \quad \lambda_3 = \left(\frac{\sqrt{2}-1}{10}\right) < 1$

$\boxed{\alpha \cdot \vec{v}_1}$

$$M^n \cdot x[0] = \alpha \cdot (\lambda_1)^n \cdot v_1 + \beta \cdot (\lambda_2)^n v_2 + \gamma \cdot (\lambda_3)^n \cdot v_3$$

Since λ_2 & λ_3 have magnitudes < 1, system will converge to $\alpha \cdot \vec{v}_1$ (steady state vector)

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

$Tx = x$

129 + 109
109
238

238 gallons of water

$$\alpha \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix} \rightarrow \alpha \begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix}$$

$\alpha = 2$

$43 + 40 + 36 = 119$

119 gallons $\cdot\, 2 = 238$

Since we start w/ 238 gallons of water & our system is conservative, starting amount of water = ending amount of water

$$\boxed{\begin{bmatrix} 86 \\ 80 \\ 72 \end{bmatrix}}$$

$86 + 80 + 72 = 238$