EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 5

This homework is due Friday, October 1, 2021, at 23:59.
Self-grades are due Monday, October 4, 2021, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Note 8 through 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. You are always welcome and encouraged to read beyond this as well.

2. Subspaces, Bases and Dimension

For each of the sets $\mathbb{U}$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(b) **(PRACTICE/OPTIONAL)** $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(d) **(PRACTICE, OPTIONAL)** $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

3. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

4. The Dynamics of Romeo and Juliet's Love Affair

Learning Goal: *Eigenvalues and eigenvectors of state transition matrices tend to reveal useful information about the dynamical systems they model. This problem serves as an example of extracting useful information through analysis of the eigenvalues of the state transition matrix of a dynamical system.*

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ denote Juliet's feelings about Romeo on day n, where $R[n]$ and $J[n]$ are **scalars**. The **sign** of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The **magnitude** (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger magnitude of $R[n]$ means that Romeo has a stronger emotion towards Juliet (strong love if $R[n] > 0$ or strong hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \ldots$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denotes the state transition matrix for our dynamic system model.

The selection of the parameters a, b, c, d results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. a, b, c, d) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a+b=c+d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1.
Show that

$$\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_2.
Hint: *Consider $\mathbf{A}\vec{v}_1$. Is it equal to a scalar multiple of $\vec{v}_1$? Repeat a similar process for $\vec{v}_2$.*

For parts (b) - (d), consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(b) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

(c) Determine all of the non-zero *steady states* of the system. That is, find all possible state vectors $\vec{s}_*$ such that if Romeo and Juliet start at, or enter, any of those state vectors, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

(e) Now suppose we have a different state-transition matrix $\mathbf{A}$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$, with a corresponding eigenvalue $\lambda_1 = 2$. Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time in this setup? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

5. Noisy Images

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Learning Goal: *The imaging lab uses the eigenvalues of the masking matrix to understand which masks are better than others for image reconstruction in the presence of additive noise. This problem explores the underlying mathematics.*

In lab, we used a single pixel camera to capture many measurements of an image $\vec{i}$. A single scalar measurement s_i is captured using a mask $\vec{h_i}$ such that $s_i = \vec{h_i}^T \vec{i}$. Many measurements can be expressed as a matrix-vector multiplication of the masks with the image, where the masks lie along the rows of the matrix.

$$\begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} = \begin{bmatrix} \vec{h}_1^T \\ \vdots \\ \vec{h}_N^T \end{bmatrix} \vec{i} \tag{1}$$

$$\vec{s} = \mathbf{H}\vec{i} \tag{2}$$

In the real world, noise, $\vec{w}$, creeps into our measurements, so instead we have,

$$\vec{s} = \mathbf{H}\vec{i} + \vec{w}. \tag{3}$$

(a) Express $\vec{i}$ in terms of $\mathbf{H}$ (or its inverse), $\vec{s}$, and $\vec{w}$. Assume $\mathbf{H}$ is invertible. (*Hint:* Think about what you did in the imaging lab.)

(b) It turns out that the eigenvalues of $\mathbf{H}$ and $\mathbf{H}^{-1}$ impact how well we can reconstruct the image from the measurements $\vec{s}$. We will see this in subsequent parts of the problem. First, let us compute the eigenvalues of $\mathbf{H}^{-1}$. The eigenvalues of $\mathbf{H}^{-1}$ are actually related to the eigenvalues of $\mathbf{H}$! Show that if λ is an eigenvalue of a matrix $\mathbf{H}$, then $\frac{1}{\lambda}$ is an eigenvalue of the matrix $\mathbf{H}^{-1}$.
Hint: Start with an eigenvalue λ and one corresponding eigenvector $\vec{v}$, such that they satisfy $\mathbf{H}\vec{v} = \lambda\vec{v}$.

(c) We are going to try different $\mathbf{H}$ matrices in this problem and compare how they deal with noise. Run all of the cells in the attached IPython notebook. Observe the **plots and the printed results.** Which matrix performs best in reconstructing the original image and why? What do you observe regarding the eigenvalues of matrices $\mathbf{H_1}, \mathbf{H_2}$ and $\mathbf{H_3}$? What special matrix is $\mathbf{H}_1$? (*Notice that each plot in the iPython notebook returns the result of trying to image a noisy image as well as the minimum absolute value of the eigenvalue of each matrix.*) Comment on the effect of small eigenvalues on the noise in the image.

(d) Now, because there is noise in our measurements, there will be noise in our recovered image. However, the noise is scaled. From the results of part (a), you know that: $\vec{i} = \mathbf{H}^{-1}\vec{s} - \mathbf{H}^{-1}\vec{w}$, so the impact of the noise on the image $\vec{i}$ is given by $\mathbf{H}^{-1}\vec{w}$.
Let us call this quantity $\hat{\vec{w}}$, often called "w-hat".

$$\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} \tag{4}$$

To analyze how this transformation alters $\vec{w}$, we represent $\vec{w}$ as a linear combination of the eigenvectors of $\mathbf{H}^{-1}$,

$$\vec{w} = \alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N, \tag{5}$$

where, $\vec{b}_i$ is $\mathbf{H}^{-1}$'s eigenvector corresponding to eigenvalue $\frac{1}{\lambda_i}$.

Show that we can express the noise in the recovered image as the following linear combination of the vectors $\vec{b}_i$:

$$\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} = \alpha_1\frac{1}{\lambda_1}\vec{b}_1 + \ldots + \alpha_N\frac{1}{\lambda_N}\vec{b}_N. \tag{6}$$

Now, if λ_i is very large, will the coefficient of $\vec{b}_i$ be large or small in $\hat{\vec{w}}$? If we want $\hat{\vec{w}}$ to be as small as possible, do we prefer large λ_i's or small λ_i's

6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.