EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 6

This homework is due Friday, October 8, 2021, at 23:59.
Self-grades are due Tuesday, October 12, 2021, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

   For this homework, please read Note 11, which introduces the basics of circuit analysis and node voltage analysis. Please also read Note 12, which introduces using circuits for modelling. You are always welcome and encouraged to read beyond this as well. **Question to answer: What is the value of having a systematic procedure for solving circuits?**

2. **Page Rank**

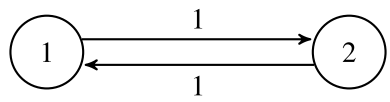
   Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

   In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

   As we have seen in lecture and discussion the "transition matrix", **T**, can be constructed using the state transition diagram, as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

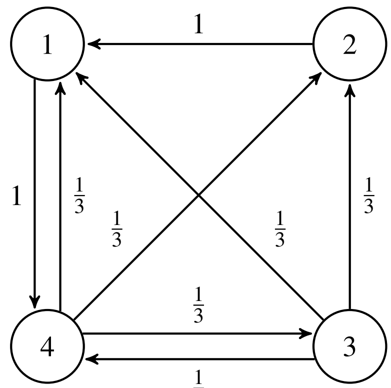
   The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

   (a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



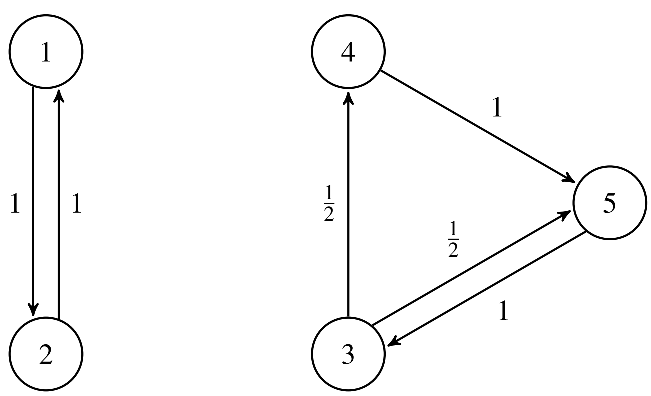
Graph A

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. Carefully read the Python documentation for `numpy.linalg.eig` to understand what this function does and what it returns. Graph B is shown below, with weights in place to help you construct the transition matrix.



Graph B

(c) Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? Again, graph C with weights in place is shown below. You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

3. Properties of Pump Systems - II

Learning Objectives: *This problem builds on the pump examples we have been doing, but is meant to help you learn to do proofs in a step by step fashion. Can you generalize intuition from a simple example?*

We consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water moved is written on top of the edge.

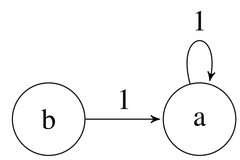


Figure 1: Pump system

We want to prove the following theorem. We will do this step by step.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(a) Rewrite the theorem statement for a graph with only two reservoirs.

(b) Since the problem does not specify the transition matrix, let us consider the transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$. Write out what is "known" or what is given to you in the theorem statement in mathematical form.
Hint: In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph.

(c) Now write out what is to be proved in mathematical form.

(d) Prove the statement for the case of two reservoirs.

(e) Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about the columns.

4. Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

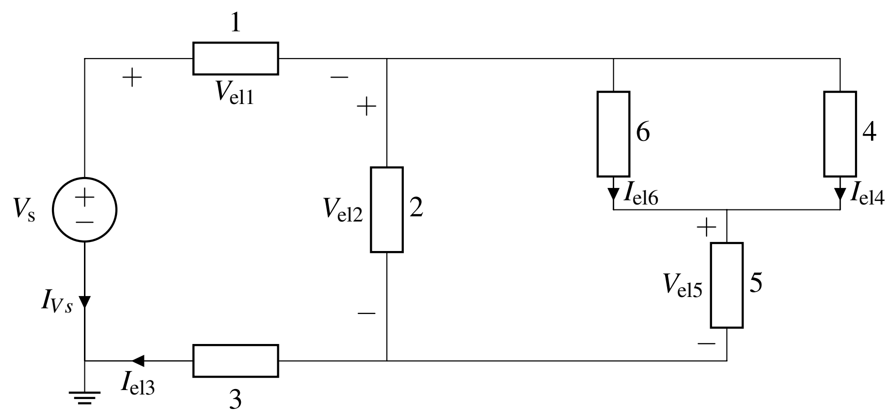
(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.
Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

(b) Let a square matrix $\mathbf{A}$ have, for each row, entries that sum to one. Show that $\vec{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices have, for each column, entries that sum to 1.

5. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KCL and KVL equations.*



(a) How many nodes does the above circuit have? Label them.
Note: The ground node has been selected for you, so you don't need to label that, but you need to include it in your node count.

(b) Notice that elements 1 - 6 and the voltage source V_s have either the *voltage across* or the *current through* them not labeled. Label the missing *voltages across* or *currents through* for elements 1 - 6, and the voltage source V_s, so that they all follow **passive sign convention**.

(c) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

(d) Write one KCL equation that involves the currents through elements 1 and 2.
*Hint: This will **not** be specific to your node labeling. Your answer may contain currents through other elements too.*

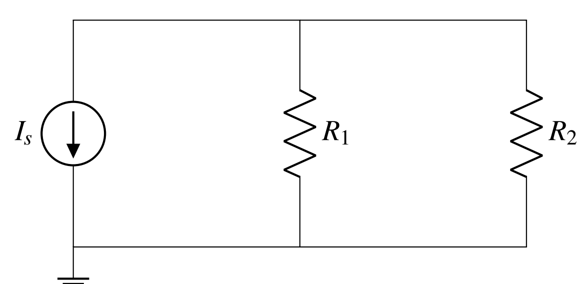
(e) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

6. Circuit Analysis

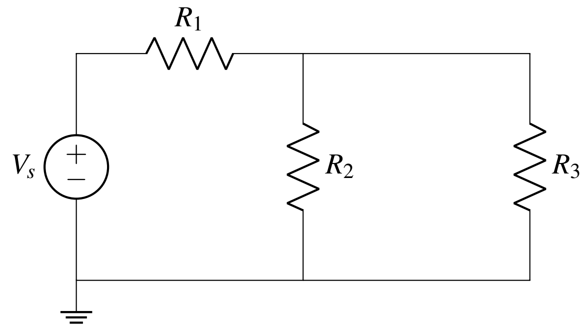
Learning Goal: *This problem will help you practice circuit analysis using the node voltage analysis (NVA) method.*

Using the steps outlined in lecture or in Note 11, analyze the following circuits to calculate the currents through each element and the voltages at each node. Use the ground node labelled for you. You may use a numerical tool such as IPython to solve the final system of linear equations.

(a) $I_s = 3\,\text{mA}, R_1 = 2\,\text{k}\Omega, R_2 = 4\,\text{k}\Omega$



(b) $V_s = 5\,\text{V}, R_1 = R_2 = 2\,\text{k}\Omega, R_3 = 4\,\text{k}\Omega$

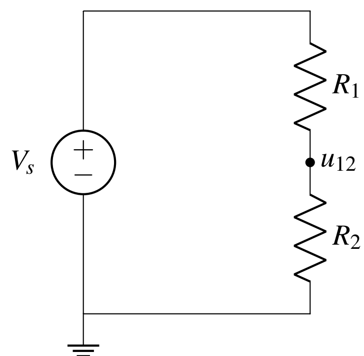


7. Voltage divider

Learning Goal: *This problem will help you practice designing circuits under given conditions using the analysis tools you've learned.*

In the following parts, $V_s = 12\,\text{V}$. **Choose resistance values such that the current through each element is $\leq$ 0.8A.**

(a) Select values for R_1 and R_2 in the circuit below such that $u_{12} = 6\,\text{V}$. This is a **design problem**, so there can be more than one set of correct answers to this problem.



8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.