
EECS 16A
Fall 2018

Designing Information Devices and Systems I

Midterm 1

Midterm 1 Solution

PRINT your student ID: _____

PRINT AND SIGN your name: _____, _____ _____
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): _____

Name and SID of the person to your left: _____

Name and SID of the person to your right: _____

Name and SID of the person in front of you: _____

Name and SID of the person behind you: _____

1. What is one of your hobbies? (2 Points)

2. Tell us about something that makes you happy. (2 Points)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

PRINT your name and student ID: _____

3. Pizza and Pirates! (18 points)

You are stuck on a deserted island and you need to find food everyday! From science class, you know that each day you need to eat:

Table 3.1: Daily doses

Food [grams]	Daily dose
Fat	4g
Carbs	2g
Protein	14g
Vitamins	6g

Thankfully, you find a pirate camp on the the island and they have 4 kinds of food; eggs, pineapple pizza, bananas, and carrots. Once again, you thank your science teacher, and remember that you know the composition of each of these foods:

Table 3.2: Food composition

Food [grams]	1 egg	1 slice of pineapple pizza	1 banana	1 carrot
Fat	1g	2g	0g	0g
Carbs	0g	2g	1g	0g
Protein	3g	3g	1g	0g
Vitamins	1g	0g	1g	1g

In order to get enough food, you decide to steal some from the pirates. But, since it is so dangerous to steal food, you want to take *exactly* what you need, no more no less. Each day, you must decide how much food to steal; number of eggs, x_e , number of pineapple pizza slices, x_p , number of bananas, x_b , and number of carrots, x_c .

- (a) (2 points) How many unknowns are there in this problem?

Solution: There are 4 unknowns. Each day, you must decide how much food to steal; number of eggs, x_e , number of pineapple pizza slices, x_p , number of bananas, x_b , and the number of carrots, x_c .

- (b) (6 points) Using Tables 3.1 and 3.2, write the equation for your daily dose of food groups in the form $\mathbf{A}\vec{x} = \vec{y}$ where $\vec{x} = [x_e, x_p, x_b, x_c]^T$. Clearly define \mathbf{A} and \vec{y} in your solution.

Solution: The four unknowns that we want to solve for are number of eggs, x_e , number of pineapple pizza slices, x_p , number of bananas, x_b , and the number of carrots, x_c . Each day, we want the amount of food we eat to be exactly equal to the daily doses in Table 3.1. We will let \vec{y} be our daily dose of each food group. To satisfy our daily needs, we need:

$$\vec{y} = \begin{bmatrix} 4 \\ 2 \\ 14 \\ 6 \end{bmatrix} \quad (1)$$

We want \mathbf{A} to map from \vec{x} to \vec{y} , or from the number of individual foods to grams of each food group. We can construct \mathbf{A} from Table 3.2.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad (2)$$

Now writing this in the form $\mathbf{A}\vec{x} = \vec{y}$:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_e \\ x_p \\ x_b \\ x_c \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 14 \\ 6 \end{bmatrix} \quad (3)$$

(c) (10 points) Now let \mathbf{A} and \vec{y} be:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 4 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{ and } \vec{y} = \begin{bmatrix} 4 \\ 2 \\ 14 \\ 6 \end{bmatrix}, \quad (4)$$

where \vec{y} is the daily dose of each food group needed.

Using the values from Equation (4), find the solution or the set of solutions for how much of each type of food you need to steal everyday, i.e. solve for \vec{x} in $\mathbf{A}\vec{x} = \vec{y}$.

Solution: First, we will construct our augmented matrix using \mathbf{A} and \vec{y} .

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 \\ 2 & 4 & 1 & 1 & 14 \\ 0 & 0 & 1 & 1 & 6 \end{array} \right]$$

We can use Gaussian elimination on our augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 \\ 2 & 4 & 1 & 1 & 14 \\ 0 & 0 & 1 & 1 & 6 \end{array} \right] \xrightarrow[R_3 - 2R_1 \rightarrow R_3]{\Leftrightarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 1 & 6 \end{array} \right] \xrightarrow[R_4 - R_3 \rightarrow R_4]{\Leftrightarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

After Gaussian elimination, our augmented matrix has a row of zeros at the bottom. This tells us that there are infinite solutions where x_c is a free variable.

We must now parameterize the set of solutions. From the augmented matrix above, we can pull out the following set of linear equations:

$$\begin{aligned} 1x_e + 2x_p &= 4 \\ 2x_p &= 2 \\ 1x_b + 1x_c &= 6 \end{aligned}$$

Let $x_b = m$ where $m \in \mathbb{R}$. Solving this set of linear equations, we get

$$\vec{x} = \begin{bmatrix} x_e \\ x_p \\ x_b \\ x_c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ m \\ 6 - m \end{bmatrix}$$

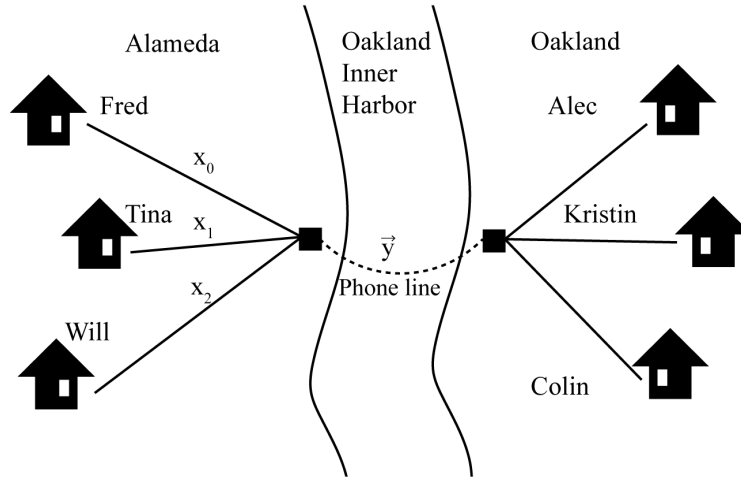
Any vector of this form is our set of possible solutions. Full credit will be given to any correct representation. You also could have represented this in terms of the free variable x_c .

$$\begin{bmatrix} 2 \\ 1 \\ 6 \\ 0 \end{bmatrix} + x_c \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

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4. Trouble in Telecomm (24 points)

Fred (x_0), Tina (x_1), and Will (x_2) each are sending messages (where each message x_0, x_1, x_2 is a real number) at the same time to Alec, Kristin, and Colin respectively.



To achieve this, the phone company will transmit \vec{y} , which is a vector of linear combinations of x_0, x_1, x_2 . Specifically,

$$\vec{y} = \mathbf{V}\vec{x} = \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}. \quad (5)$$

\mathbf{V} is the encoding matrix.

On the receiver side, Alec, Kristin and Colin need to recover x_0, x_1, x_2 respectively from \vec{y} . You are helping the phone company evaluate different choices for the columns \vec{c}_0, \vec{c}_1 and \vec{c}_2 of matrix \mathbf{V} :

$$\mathbf{V}_0 = \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix} \quad (6)$$

$$\mathbf{V}_1 = \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- (a) (8 points) You decide to characterize \mathbf{V}_0 in terms of its null space. Find a basis for the nullspace of \mathbf{V}_0 .
Solution: Analyzing the first set of codes, we can find the null space by setting up the following matrix equation:

$$\mathbf{V}_0 \vec{x} = \vec{0} \quad (7)$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \quad (8)$$

$$\underbrace{-2R_1 + R_2 \rightarrow R_2}_{\Rightarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \quad (9)$$

$$\underbrace{R_2/3 \rightarrow R_2}_{\Rightarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \quad (10)$$

$$\underbrace{-2R_2 + R_3 \rightarrow R_3}_{\Rightarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \quad (11)$$

Now we identify c is a free variable and express the other variables in terms of it.

$$b = -2c \quad (12)$$

$$a = -2c \quad (13)$$

Finally we construct the null space,

$$N(\mathbf{V}_0) = \text{span} \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\} \quad (14)$$

- (b) (8 points) If the matrix $\mathbf{V}_0 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix}$ is invertible, find its inverse. If it is not invertible, why not?

Given this, is \mathbf{V}_0 a good encoding matrix to use? Justify your answer.

Solution: The matrix \mathbf{V}_0 is non-invertible. This is because the columns are not linearly independent (twice the second plus the first is the third). This can also be identified from the null space. The null space of the matrix contains more than just the zero vector. This implies there will be an infinite number of solutions and that Alec, Kristin, and Colin will not be able to uniquely decode the sent messages. This makes \mathbf{V}_0 an unsuitable encoding matrix for the telecommunication company.

- (c) (8 points) If the matrix $\mathbf{V}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible, find its inverse. If it is not invertible, why not?

Given this, is \mathbf{V}_1 a good encoding matrix to use? Justify your answer.

Solution: The columns are linearly independent and therefore the matrix inverse exists. To find the inverse, we start by writing the matrix equation:

$$\mathbf{V}_1 \mathbf{V}_1^{-1} = \mathbf{I} \quad (15)$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{V}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (16)$$

Then we setup the augmented matrix:

$$[\mathbf{V}_1 \mid \mathbf{I}] \quad (17)$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad (18)$$

$$\begin{array}{l} R_3 \rightarrow R_1 \\ R_1 \rightarrow R_2 \\ R_2 \rightarrow R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array} \right] \quad (19)$$

$$\begin{array}{l} R_3 - R_1 \rightarrow R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \end{array} \right] \quad (20)$$

$$\begin{array}{l} R_2 + R_3 \rightarrow R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array} \right] \quad (21)$$

$$\begin{array}{l} \frac{1}{2}R_3 \rightarrow R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array} \right] \quad (22)$$

$$\begin{array}{l} R_2 - R_3 \rightarrow R_2 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array} \right] \quad (23)$$

$$\begin{array}{l} R_1 - R_2 \rightarrow R_1 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array} \right] \quad (24)$$

Reading off the right half of the augmented matrix, we get:

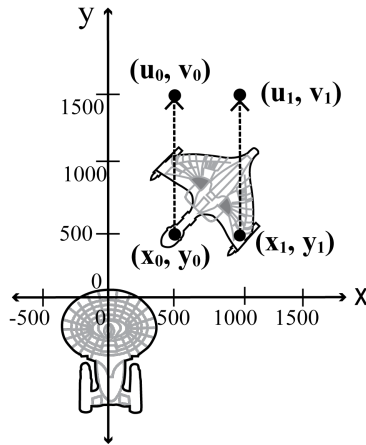
$$\mathbf{V}_1^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad (25)$$

Because the inverse exists, Alec, Kristin, and Colin will be able to uniquely decode their intended signals, thus, \mathbf{V}_1 is a suitable encoding matrix for the telecommunication company.

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5. The Romulan Ruse (32 points) While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

- (a) (6 points) The Romulan illusion technology causes a point (x_0, y_0) to transform or *map* to (u_0, v_0) . Similarly, (x_1, y_1) is mapped to (u_1, v_1) . Figure 5.1 and Table 5.1 show two points on a Romulan ship and the corresponding *mapped* points.



Original Point	Mapped Point
$(x_0, y_0) = (500, 500)$	$(u_0, v_0) = (500, 1500)$
Original Point	Mapped Point
$(x_1, y_1) = (1000, 500)$	$(u_1, v_1) = (1000, 1500)$

Table 5.1: Original and Mapped Points

Figure 5.1: Figure for part (a)

Find a transformation matrix \mathbf{A}_0 such that

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ and } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Solution: Let us assume $\mathbf{A}_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence for point (x_0, y_0) , we have:

$$\begin{bmatrix} 500 \\ 1500 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

i.e.

$$a + b = 1; \tag{26}$$

$$c + d = 3. \tag{27}$$

Similarly, for point (x_1, y_1) , we have

$$\begin{bmatrix} 1000 \\ 1500 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1000 \\ 500 \end{bmatrix} \implies \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

i.e.

$$2a + b = 1; \tag{28}$$

$$2c + d = 3. \tag{29}$$

Solving Equations (26) and (28) for a and b , we have:

$$a = 1, \text{ and } b = 0.$$

Solving Equations (27) and (29) for c and d , we have:

$$c = 0, \text{ and } d = 3.$$

Substituting values of a , b , c , and d , we have

$$\mathbf{A}_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Additionally, it can be observed from Figure 5.1 that the mapped vectors are derived by scaling the original vectors by 3 in the y-direction and by unity in the x-direction. Using Figure 5.1 and Table 5.1, we can write

$$u_0 = x_0, \text{ and } v_0 = 3y_0, \quad (30)$$

and

$$u_1 = x_1, \text{ and } v_1 = 3y_1. \quad (31)$$

Writing equations 30 and 31 in matrix-vector product form, we have

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix};$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Hence

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \quad (32)$$

- (b) (6 points) In this scenario, every point on the Romulan ship (x_m, y_m) is mapped to (u_m, v_m) , such that vector $\begin{bmatrix} x_m \\ y_m \end{bmatrix}$ is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 5.2.

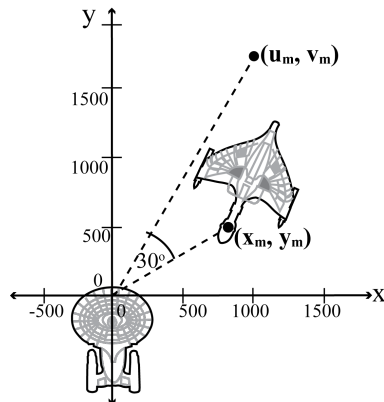


Figure 5.2: Figure for part (b)

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Table 5.2: Trigonometric Table

Find a transformation matrix \mathbf{R} such that $\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \mathbf{R} \begin{bmatrix} x_m \\ y_m \end{bmatrix}$.

Solution: Transformation matrix that rotates a vector counterclockwise by 30° is:

$$\mathbf{R}_\theta = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Transformation matrix that rotates a vector counterclockwise by 30° and scales by 2 is:

$$\mathbf{R} = 2\mathbf{R}_\theta = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

Alternatively, the transformation matrix can be written as:

$$\mathbf{R} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point $(0,0)$ towards the probe.

- (c) (10 points) The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix \mathbf{A}_p :

$$\mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

They positioned the probe at (x_p, y_p) so that it maps to

$$(u_p, v_p) = (0,0), \text{ where } \begin{bmatrix} u_p \\ v_p \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}.$$

This scenario is shown in Figure 5.3. The initial position of the torpedo is $(0,0)$ and the torpedo cannot be fired on its initial position! Impressive trick indeed!

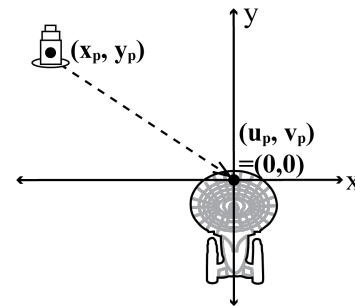


Figure 5.3: Figure for part (c)

Find the possible positions of the probe (x_p, y_p) so that $(u_p, v_p) = (0,0)$.

Solution: We need to solve for

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So essentially we need to find the nullspace of the matrix \mathbf{A}_p . Using Gaussian Elimination on the augmented matrix, we have:

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow x_p + 3y_p = 0 \Rightarrow x_p = -3y_p.$$

The solution is $\alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where α is $\{\alpha \in \mathbb{R}\}$. So $\begin{bmatrix} x_p \\ y_p \end{bmatrix}$ should be in the span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Alternatively, any point (x_p, y_p) that is on the line: $x = -3y$, would represent all possible positions of the probe.

- (d) (10 points) It turns out the Romulan engineers were not as smart the Enterprise engineers. Their calculations did not work out and they positioned the probe at (x_q, y_q) such that the *cloaking* (transformation) matrix, \mathbf{A}_p , mapped it to (u_q, v_q) , where

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix}, \text{ and } \mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

As a result, the torpedo while traveling along a straight line from $(0,0)$ to (u_q, v_q) , hit the probe at (x_q, y_q) on the way!

The scenario is shown in Figure 5.4. For the torpedo to hit the probe, we must have $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, where λ is a real number.

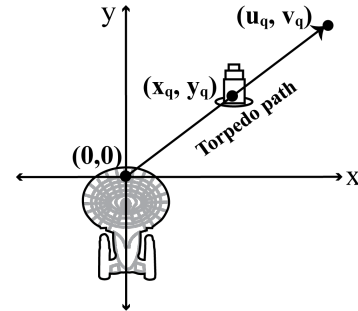


Figure 5.4: Figure for part (d)

Find the possible positions of the probe (x_q, y_q) so that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$. Remember that the torpedo cannot be fired on its initial position $(0,0)$.

Solution: We need to solve for $\mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, i.e. we need to find the eigenvectors of \mathbf{A}_p . Let's start by finding the eigenvalues:

$$\det \left\{ \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right\} = 0$$

$$\det \left\{ \begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix} \right\} = 0$$

So we have the characteristic polynomial:

$$(1-\lambda)(6-\lambda) - (3)(2) = 0$$

$$\Rightarrow \lambda = 0, 7$$

Using $\lambda = 0$, we have: $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which will map (x_q, y_q) to the original position of the torpedo. The torpedo cannot be fired on its original position. So $\lambda = 0$ will not provide a valid solution.

Using $\lambda = 7$, we have:

$$(\mathbf{A}_p - 7\mathbf{I}) \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using Gaussian Elimination on the augmented matrix form, we have

$$\left[\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow 2x_q - y_q = 0 \Rightarrow y_q = 2x_q$$

The solution is $\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where α is $\{\alpha \in \mathbb{R} : \alpha \neq 0\}$. So $\begin{bmatrix} x_q \\ y_q \end{bmatrix}$ should be in the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Alternatively, any point (x_q, y_q) that is on the line: $y = 2x$, excluding $(0,0)$, would represent all possible positions of the probe.

PRINT your name and student ID: _____

6. A Tropical Tale of Triumph: Does Pineapple Come Out on Top? (52 points)

(Based on a true story) During a discussion section, one of your TAs, Nick, makes the claim that pineapple belongs on pizza. Another TA, Elena, strongly disagrees. Naturally, a war starts and students begin to flock to the TA they agree with, switching discussion sections every week. Some students don't have an opinion and go to Lydia's section since she is neutral in the matter. As a 16A student, you want to analyze this war to see how it will play out.

- (a) (6 points) You manage to capture the behavior of the students as a transition matrix, but want to visualize it. You've written out the transition matrix \mathbf{M} :

$$\mathbf{M} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 1 \\ 0.25 & 0.5 & 0 \end{bmatrix}$$

such that

$$\begin{bmatrix} x_{\text{Elena}}[n+1] \\ x_{\text{Nick}}[n+1] \\ x_{\text{Lydia}}[n+1] \end{bmatrix} = \mathbf{M} \begin{bmatrix} x_{\text{Elena}}[n] \\ x_{\text{Nick}}[n] \\ x_{\text{Lydia}}[n] \end{bmatrix}.$$

Each element of the state vector $\vec{x}[n] = [x_{\text{Elena}}[n] \ x_{\text{Nick}}[n] \ x_{\text{Lydia}}[n]]^T$ represents the number of students attending that section at timestep n . **Fill in values in the boxes in Figure 6.1 below** such that the diagram represents the transition matrix \mathbf{M} .

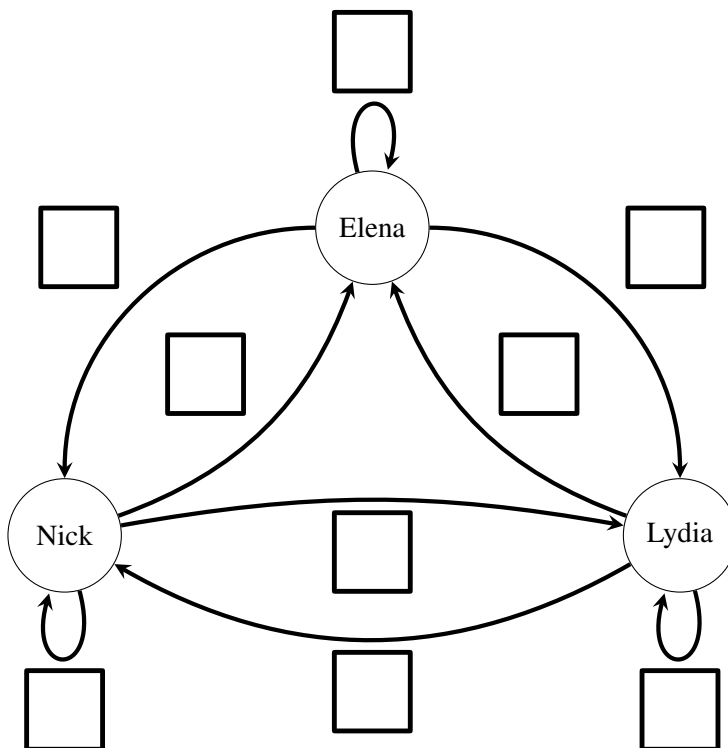


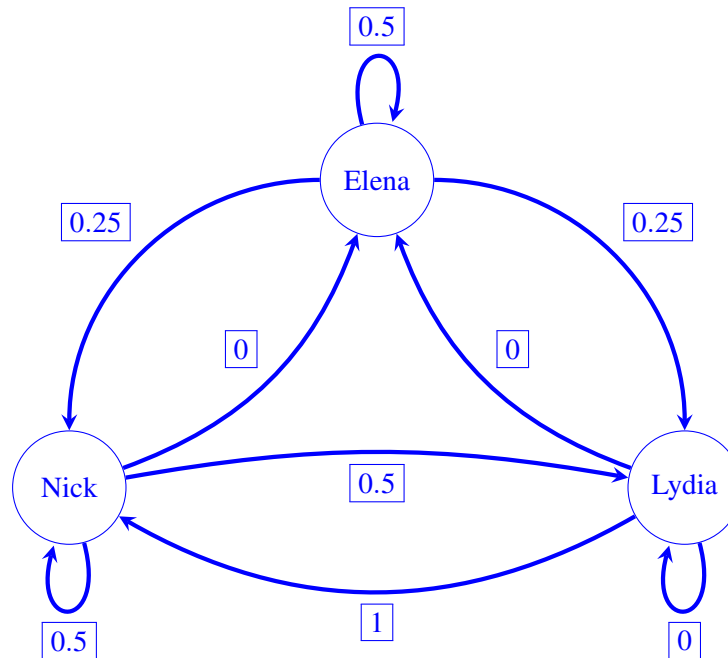
Figure 6.1: Flow diagram for discussion sections

Solution: Writing out the system of equations,

$$\begin{aligned}x_{\text{Elena}}[t+1] &= \frac{1}{2}x_{\text{Elena}}[t] \\x_{\text{Nick}}[t+1] &= \frac{1}{4}x_{\text{Elena}}[t] + \frac{1}{2}x_{\text{Nick}}[t] + x_{\text{Lydia}}[t] \\x_{\text{Lydia}}[t+1] &= \frac{1}{4}x_{\text{Elena}}[t] + \frac{1}{2}x_{\text{Nick}}[t]\end{aligned}$$

It helps to write these out so you don't accidentally use the transpose!

Filling in the diagram from above:



- (b) (10 points) Your friend Vlad tells you that your transition matrix \mathbf{M} was wrong, and gives you a new transition matrix \mathbf{S} , **which has a steady state**. In order to find who wins the war, you need to find how many students end up in each section after everything has settled. **Find a vector \vec{x} that represents a steady state of \mathbf{S} .**

$$\mathbf{S} = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 1 \end{bmatrix}$$

Solution: To find the eigenvector corresponding to $\lambda = 1$, use the equation relating eigenvalues to eigenvectors: $\mathbf{S}\vec{x} = \lambda\vec{x}$ and substitute in 1 for λ .

$$\begin{aligned}\mathbf{S}\vec{x} &= \lambda\vec{x} \\ &= \vec{x} \leftarrow \lambda = 1 \\ \mathbf{S}\vec{x} - \vec{x} &= \vec{0} \\ (\mathbf{S} - \mathbf{I})\vec{x} &= \vec{0}\end{aligned}$$

To find the eigenvector, we must solve for the vector \vec{x} which satisfies the above equation. In other words, we need to find $\text{Null}(\mathbf{S} - \mathbf{I})$.

$$\begin{aligned}\mathbf{S} - \mathbf{I} &= \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.8 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0.3 & 0 & 0 \end{bmatrix}\end{aligned}$$

We do this using Gaussian elimination

$$\begin{aligned}\begin{bmatrix} -0.8 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0.3 & 0 & 0 \end{bmatrix} &\xrightarrow{R_1 + R_3 \mapsto R_1} \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0.3 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{\text{switch } R_1, R_3} \begin{bmatrix} 0.3 & 0 & 0 \\ 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \end{bmatrix} \\ &\xrightarrow{\frac{10}{3}R_1 \mapsto R_1} \\ &\xrightarrow{2R_2 \mapsto R_2} \\ &\xrightarrow{R_3 + R_2 \mapsto R_3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{R_2 - R_1 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

Rearranging the equations from our upper triangular matrix above, we get the following:

$$\begin{aligned}x_1 &= 0 \\ -x_2 &= 0\end{aligned}$$

Setting x_3 as a free variable, we find

$$\text{Null}(\mathbf{S} - \mathbf{I}) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

where the eigenvector \vec{x} is any vector contained within this space. During the exam, the clarification was added to ask for a nonzero vector. Any nonzero scalar multiple of $[0 \ 0 \ 1]^T$ is a correct solution.

- (c) (6 points) Your other friend Gireeja points out that the arguments are causing new people to join the sections and others to leave entirely. In other words, the system is not conservative! The new system

can be modeled with a state transition matrix \mathbf{A} that has the following eigenvalue/eigenvector pairings:

$$\lambda_1 = 1 : \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} : \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2 : \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You want the number of students in sections to stabilize. Which of the vectors below represent **steady states** of the system, i.e. \vec{x} such that $\mathbf{A}\vec{x} = \vec{x}$? **Fill in the circle(s) to the left of these vector(s).**

$$\begin{array}{cccc} \circ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\ \circ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix} \end{array}$$

Solution: The state \vec{x} can be written as a linear combination of the eigenvectors:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

Applying \mathbf{A} to \vec{x} ,

$$\mathbf{A}\vec{x} = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3$$

$$\vec{x} = c_1 \vec{v}_1 + \frac{1}{2} c_2 \vec{v}_2 + 2c_3 \vec{v}_3 \leftarrow \mathbf{A}\vec{x} = \vec{x}$$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = c_1 \vec{v}_1 + \frac{1}{2} c_2 \vec{v}_2 + 2c_3 \vec{v}_3$$

Given the condition that we need a steady state and because \vec{v}_1 , \vec{v}_2 , and \vec{v}_3 are linearly independent, we know that

- $c_2 = c_3 = 0$
- $c_1 \geq 0$. If $c_1 = 0$, this means there are no students to begin with, which makes for a very boring (but extremely stable!) war. If $c_1 > 0$, it just satisfies the eigenvalue equation $\mathbf{A}c_1 \vec{v}_1 = c_1 \vec{v}_1$.

Long story short, we're looking for scaled versions of $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{array}{cccc} \otimes \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\ \circ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \otimes \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix} \end{array}$$

(d) (6 Points) Assume we are still working with the same state transition matrix \mathbf{A} as in part (c). Which of the vectors below represent **initial states** such that the number of students in the sections keeps growing? **Fill in the circle(s) to the left of these vector(s).**

$$\begin{array}{cccc} \circ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\ \circ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix} \end{array}$$

Solution: Writing $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, we want to see what situations down the line will never lead to a steady state.

$$\begin{aligned} \mathbf{A}^n \vec{x} &= \lambda_1^n c_1 \vec{v}_1 + \lambda_2^n c_2 \vec{v}_2 + \lambda_3^n c_3 \vec{v}_3 \\ &= c_1 \vec{v}_1 + \frac{1}{2^n} c_2 \vec{v}_2 + 2^n c_3 \vec{v}_3 \end{aligned}$$

After waiting an infinite amount of time, i.e. $n \rightarrow \infty$, the \vec{v}_1 component will stay steady; the \vec{v}_2 component will go to zero; and the \vec{v}_3 component will explode to infinity unless $c_3 = 0$. Thus, in order for things to explode, the following conditions must be true:

- $c_3 > 0$
- c_2 and c_1 can be any nonnegative (including zero) value

In other words, we want vectors which have a nonzero $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ component. This causes the system to never settle into a steady state, i.e. there will always be people being drawn into the flame war of pineapples on pizza.

$$\begin{array}{cccc} \circ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\ \circ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \otimes \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix} \end{array}$$

Note: Had there been an eigenvalue of -1 with an associated eigenvector \vec{v}_4 , applying \mathbf{A} to \vec{v}_4 repeatedly would just lead to oscillation between the positive and negative vectors. This is another form of non-convergence.

- (e) (6 points) Again assume we are still working with the same state transition matrix \mathbf{A} as in part (c). Which of the vectors below represent **initial states** such that everyone leaves the system, i.e. $\lim_{n \rightarrow \infty} \mathbf{A}^n \vec{x} = \vec{0}$? **Fill in the circle(s) to the left of these vector(s).**

$$\begin{array}{cccc} \circ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\ \circ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix} \end{array}$$

Solution: Defining x and finding $\mathbf{A}\vec{x}$:

$$\begin{aligned} \vec{x} &= c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \\ \mathbf{A}^n \vec{x} &= \lambda_1^n c_1 \vec{v}_1 + \lambda_2^n c_2 \vec{v}_2 + \lambda_3^n c_3 \vec{v}_3 \end{aligned}$$

After waiting an infinite amount of time, i.e. $n \rightarrow \infty$, the \vec{v}_1 component will stay steady; the \vec{v}_2 component will go to zero; and the \vec{v}_3 component will explode to infinity unless $c_3 = 0$.

- $c_1 = 0$ so there are no nonzero steady states.
- $c_3 = 0$ so the system does not explode.
- $c_2 \geq 0$

We know that after a long time, any state that is a scaled form of \vec{v}_2 associated with eigenvalue $\lambda_2 = \frac{1}{2}$ will approach $\vec{0}$, i.e. the number of people will keep halving until there are none left.

tl;dr, we're looking for any scaled version of $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$\begin{array}{cccc} \circ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \circ \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix} & \otimes \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \\ \circ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix} & \circ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \otimes \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix} \end{array}$$

- (f) (16 Points) Let us generalize the idea of convergence. Consider the following system:

$$\vec{x}[n+1] = \mathbf{T}\vec{x}[n]$$

where \vec{x} is a vector with N elements and \mathbf{T} is any $N \times N$ matrix unrelated to the previous parts. \mathbf{T} has N distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$, and N associated eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N$ such that $\mathbf{T}\vec{v}_i = \lambda_i \vec{v}_i$ for $1 \leq i \leq N$. Let $|\lambda_1| > 1$. Prove that there exists at least one initial state $\vec{x}[0]$ for this system such that it does not converge to a steady state.

Solution: We'll generalize in terms of i , and then set $i = 1$. Let \vec{v}_i be a nonzero eigenvector corresponding to the eigenvalue λ_i . We know that such a vector must exist because $\det(\mathbf{T} - \lambda_i \mathbf{I}) = 0$. Also, let $\vec{x}[0] = \vec{v}_i$, and consider $\lim_{n \rightarrow \infty} \mathbf{T}^n \vec{x}[0]$.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbf{T}^n \vec{x}[0] &= \lim_{n \rightarrow \infty} \mathbf{T}^n \vec{v}_i \\
&= \lim_{n \rightarrow \infty} \lambda_i^n \vec{v}_i \\
&= \left(\lim_{n \rightarrow \infty} \lambda_i^n \right) \vec{v}_i
\end{aligned}$$

However, since $|\lambda_i| > 1$, $\lim_{n \rightarrow \infty} |\lambda_i^n| \rightarrow \infty$. Thus, the system does not converge to a steady state, and \vec{v}_i (or any scalar multiple of it) is an initial state such that the system does not converge.

Common Mistakes

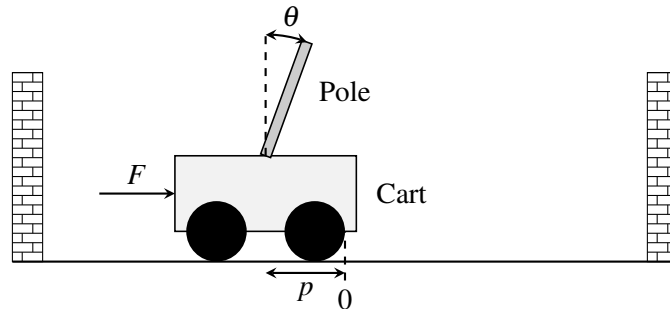
- Proofs which use diagonalization must prove that \mathbf{T} is diagonalizable. In this case it is, since there are N distinct eigenvalues with N respective associated eigenvectors, the N eigenvectors are all linearly independent, implying the matrix is diagonalizable.
 - Note that the problem states that $|\lambda| > 1$. Missing the case in which $\lambda < -1$ and only addressing the scenario in which $\lambda > 1$ proves only that $\lambda > 1$ is a case for non-convergence and not the more general case of $|\lambda| > 1$.
 - Many students who used diagonalization asserted that if the first component of $\vec{x}[0]$ is nonzero, then the system would not converge. The correct condition is if $\mathbf{V}^{-1}\vec{x}[0]$ has a nonzero first component.
 - When choosing $\vec{x}[0] = \sum_j \alpha_j \vec{v}_j$, it needs to be stated that $\alpha_1 \neq 0$
- (g) (2 points) Does pineapple belong on pizza? (*Hint: Full points for honesty!*)
Solution: Whether your answer is correct or not depends on if Elena or Nick graded your exam. All answers were given full credit though!

PRINT your name and student ID: _____

7. A Balancing Act! (46 points)

Your friend has started a San Francisco tour business using segways and she needs your help to find which sensor to use for her segways.

We model the segway as a cart-pole system:



A cart-pole system can be fully described at any time step n by its position $p[n]$, velocity $\dot{p}[n]$, angle $\theta[n]$, and angular velocity $\dot{\theta}[n]$. We write this as a “state vector”:

$$\vec{x}[n] = \begin{bmatrix} p[n] \\ \dot{p}[n] \\ \theta[n] \\ \dot{\theta}[n] \end{bmatrix}$$

In this case, the cart-pole system can be represented by the following linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n], \quad (33)$$

where $\mathbf{A} \in \mathbb{R}^{4 \times 4}$.

Since no sensor can measure all four elements of the state vector, the following model can be used to represent the measurements made by a certain sensor:

$$\vec{y}[n] = \mathbf{C}\vec{x}[n], \quad (34)$$

where $\mathbf{C} \in \mathbb{R}^{2 \times 4}$ expresses the measurement matrix of the sensor. Each sensor has an output: $\vec{y}[n] \in \mathbb{R}^2$.

We have a few different sensors and we need to determine if the initial state $\vec{x}[0]$ can be uniquely identified using each sensor.

(a) (4 points) Express $\vec{y}[0]$ in terms of $\vec{x}[0]$.

Solution: From Equation (34), we get (by substituting $n = 0$):

$$\vec{y}[0] = \mathbf{C}\vec{x}[0] \quad (35)$$

(b) (6 points) Express $\vec{x}[1]$ and $\vec{y}[1]$ in terms of $\vec{x}[0]$.

Solution:

From Equation (33), we get (by substituting $n = 0$):

$$\vec{x}[1] = \mathbf{A}\vec{x}[0] \quad (36)$$

Using the result for $\vec{x}[1]$ and applying Equation (34), we get the following:

$$\vec{y}[1] = \mathbf{CA}\vec{x}[0] \quad (37)$$

(c) (10 points) The first sensor measures $p[n]$ and $\theta[n]$, and has the following measurement matrix:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For this part, use the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now:

- For any $\vec{x}[0]$ we have $\vec{y}[0] \in \mathbb{R}^2$ and $\vec{y}[1] \in \mathbb{R}^2$.
- Stacking $\vec{y}[0]$ and $\vec{y}[1]$ on top of each other gives a 4×1 column vector: $\vec{z} = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} \in \mathbb{R}^4$.
- Expressing $\vec{y}[0]$ and $\vec{y}[1]$ in terms of $\vec{x}[0]$, we can write a system of equations:

$$\mathbf{Q}\vec{x}[0] = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} = \vec{z}, \quad (38)$$

where $\mathbf{Q} \in \mathbb{R}^{4 \times 4}$, $\vec{z} \in \mathbb{R}^4$ and $\vec{x}[0] \in \mathbb{R}^4$.

If your friend can recover $\vec{x}[0]$ from any given \vec{z} , calculate $\vec{x}[0]$ for $\vec{y}[0] = \begin{bmatrix} 5.0 \\ 0.1 \end{bmatrix}$, and $\vec{y}[1] = \begin{bmatrix} 6.0 \\ 0.2 \end{bmatrix}$. If $\vec{x}[0]$ cannot be recovered, explain why.

Solution:

Taking the expressions derived previously for $\vec{y}[0]$ and $\vec{y}[1]$, and using the given matrices \mathbf{C} and \mathbf{A} in this part, we can express the system of equations using a single matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \end{bmatrix}$$

For your friend to recover $\vec{x}[0]$ from any given \vec{z} , we need that the solution to Equation 38 is unique. For uniqueness, we can see the rows of \mathbf{Q} are linearly independent, so that when we apply Gaussian elimination, we will have four pivots. This means that we should have a unique solution, $\vec{x}[0]$, to Equation 38 for a given \vec{z} .

Thus we have shown that we can recover $\vec{x}[0]$ for any given \vec{z} . We now perform Gaussian elimination using the given value of \vec{z} on the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5.0 \\ 0 & 0 & 1 & 0 & 0.1 \\ 1 & 0.5 & 0 & 0 & 6.0 \\ 0 & 0 & 1 & 0.5 & 0.2 \end{array} \right]$$

The first set of row operations we use is:

- $2(R3 - R1) \rightarrow R3$
- $2(R4 - R2) \rightarrow R4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5.0 \\ 0 & 0 & 1 & 0 & 0.1 \\ 0 & 1 & 0 & 0 & 2.0 \\ 0 & 0 & 0 & 1 & 0.2 \end{array} \right]$$

Finally, we swap row 2 and row 3 to get an upper triangular matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5.0 \\ 0 & 1 & 0 & 0 & 2.0 \\ 0 & 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 & 0.2 \end{array} \right]$$

From this, we find the solution,

$$\vec{x}[0] = \begin{bmatrix} 5.0 \\ 2.0 \\ 0.1 \\ 0.2 \end{bmatrix}$$

(d) (10 points) The second sensor measures $p[n]$ and $\theta[n]$, and has the following measurement matrix:

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For this part, use the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Again:

- For any $\vec{x}[0]$ we have $\vec{y}[0] \in \mathbb{R}^2$ and $\vec{y}[1] \in \mathbb{R}^2$.
- Stacking $\vec{y}[0]$ and $\vec{y}[1]$ on top of each other gives a 4×1 column vector: $\vec{z} = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} \in \mathbb{R}^4$.
- Expressing $\vec{y}[0]$ and $\vec{y}[1]$ in terms of $\vec{x}[0]$, we can write a system of equations:

$$\mathbf{Q}\vec{x}[0] = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} = \vec{z}, \quad (39)$$

where $\mathbf{Q} \in \mathbb{R}^{4 \times 4}$, $\vec{z} \in \mathbb{R}^4$ and $\vec{x}[0] \in \mathbb{R}^4$.

If your friend can recover $\vec{x}[0]$ from any given \vec{z} , calculate $\vec{x}[0]$ for $\vec{y}[0] = \begin{bmatrix} 2.0 \\ 0.2 \end{bmatrix}$, and $\vec{y}[1] = \begin{bmatrix} 2.0 \\ 0.2 \end{bmatrix}$. If $\vec{x}[0]$ cannot be recovered, explain why.

Solution:

Taking the expressions derived previously for $\vec{y}[0]$ and $\vec{y}[1]$, and using the given matrices \mathbf{C} and \mathbf{A} in this part, we can express the system of equations using a single matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We use the same criteria as in part (c): we want to check if the solution to Equation 39 is unique.

We see that row 3 and row 4 are copies of row 1 and row 2, respectively. This means that \mathbf{Q} has only two linearly independent rows. When we perform Gaussian elimination, there will be only two pivots - we need four pivots to determine $\vec{x}[0]$ uniquely.

So we cannot recover $\vec{x}[0]$ from any given \vec{z} .

- (e) (16 points) The third sensor measures $p[n]$ and $\theta[n]$, and has the following measurement matrix:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For this part, use the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now:

- For n measurements, we have $\vec{y}[0] \in \mathbb{R}^2$, $\vec{y}[1] \in \mathbb{R}^2$, ..., $\vec{y}[n-1] \in \mathbb{R}^2$.
- Stacking the measurements $\vec{y}[0], \vec{y}[1], \dots, \vec{y}[n-1]$ on top of each other gives a $2n \times 1$ column vector:

$$\vec{z} = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \\ \dots \\ \vec{y}[n-1] \end{bmatrix} \in \mathbb{R}^{2n}. \quad (40)$$

- Expressing $\vec{y}[0], \vec{y}[1], \dots, \vec{y}[n-1]$ in terms of $\vec{x}[0]$, we can write the system of equations as:

$$\mathbf{Q}\vec{x}[0] = \vec{z}, \quad (41)$$

where $\mathbf{Q} \in \mathbb{R}^{2n \times 4}$, $\vec{z} \in \mathbb{R}^{2n}$ and $\vec{x}[0] \in \mathbb{R}^4$.

What is the minimum number of measurements needed to recover $\vec{x}[0]$ from \vec{z} ? Show the work to justify your answer.

Solution: Similar to parts (c) and (d), to uniquely recover $\vec{x}[0]$ from \vec{z} , we need to ensure that the solution to Equation 41 is unique.

Our goal, then, is to pick the smallest value of n so that we have a unique solution $\vec{x}[0]$ for any choice of \vec{z} .

n = 1: In this case, we only measure $\vec{y}[0]$ and have the following for \mathbf{Q} :

$$\mathbf{Q} = [\mathbf{C}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We do not have enough information to find $\vec{x}[0]$, as there are only two pivots in \mathbf{Q} - we need four pivots to uniquely identify $\vec{x}[0]$.

n = 2: In this case, we measure $\vec{y}[0]$ and $\vec{y}[1]$, resulting in the following for \mathbf{Q} :

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that row 2 and row 4 are identical. This implies that, after performing Gaussian elimination, we will have only three pivots. Again, we need four pivots to uniquely identify $\vec{x}[0]$.

n = 3: In this case, we measure $\vec{y}[1]$, $\vec{y}[1]$, and $\vec{y}[2]$, resulting in the following for \mathbf{Q} :

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case, we can see that rows 1,2,3,5 are linearly independent. We have four linearly independent rows, which suggests we have four pivots when doing Gaussian Elimination. So if a solution to Equation 41 exists, it will be unique.

Therefore, the minimum number of measurements needed is 3.