

Midterm 2 Solution
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**1. HONOR CODE**

If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

*I will respect my classmates and the integrity of this exam by following this honor code. I affirm:*

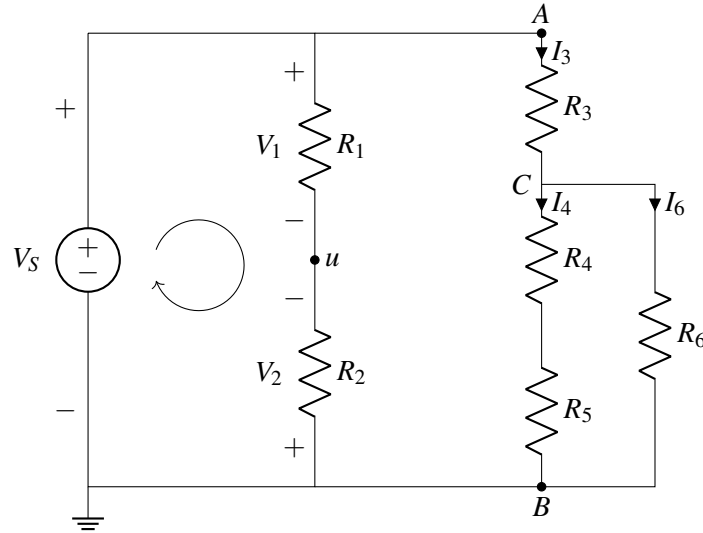
- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my two reference cheat sheets.*
- *I did not collaborate with any other human being on this exam.*

2. (a) **What other courses are you taking this semester?** *All answers will be awarded full credit; you can be brief. (2 points)*
- (b) **What has been your favorite part of 16A so far?** *All answers will be awarded full credit; you can be brief. (2 points)*

### 3. Circuit Analysis (18 points)

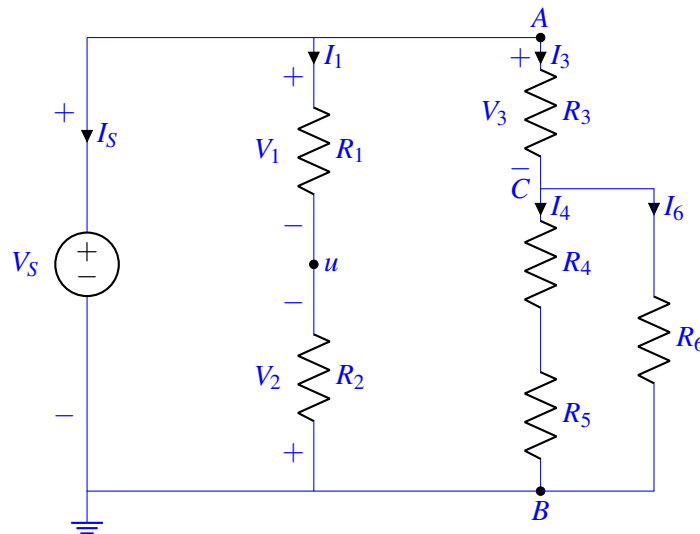
For the circuit in the following diagram, answer parts (a) – (e).

You should **not** change the labels that are already given in the diagram. If you add any additional labels for your analysis, you should show your labels in the answer sheet for the corresponding part(s).



- (a) (3 points) Redraw the circuit diagram in your answer sheet. Following the passive sign convention, **label** (i) the current  $I_S$  through the voltage source  $V_S$ , (ii) the current  $I_1$  through the resistor  $R_1$ , and (iii) the voltage  $V_3$  across the resistor  $R_3$ .

**Solution:** Following the passive sign convention (current flows into the terminal with a positive voltage), we have the labels for  $I_S$ ,  $I_1$ , and  $V_3$  shown in the following diagram.



- (b) (2 points) **Write the KVL expression** for the loop drawn in the circuit diagram in terms of voltages  $V_S$ ,  $V_1$ , and  $V_2$ .

**Solution:** If we travel in the loop, we will first meet the negative terminal of  $V_S$ , the positive terminal of  $V_1$ , and the negative terminal of  $V_2$ , respectively. So the KVL expression is

$$-V_S + V_1 - V_2 = 0.$$

Any equivalent expressions (for example,  $V_S - V_1 + V_2 = 0$ ,  $V_S + V_2 = V_1$ , etc.) are acceptable.

- (c) (2 points) **Write the KCL expression** at node  $C$  in terms of currents  $I_3$ ,  $I_4$ , and  $I_6$  as labeled in the circuit diagram.

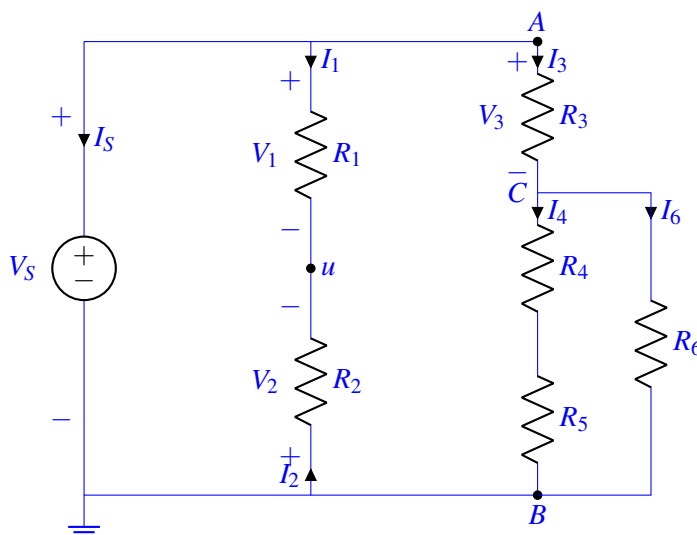
**Solution:**  $I_3$  flows into the node, and  $I_4$ ,  $I_6$  flow out of the node, so the KCL expression is

$$-I_3 + I_4 + I_6 = 0.$$

Any equivalent expressions (for example,  $I_3 - I_4 - I_6 = 0$ ,  $I_3 = I_4 + I_6$ , etc.) are acceptable.

- (d) (5 points) Given  $V_S = 5$  V,  $R_1 = 1$  k $\Omega$ ,  $R_2 = 4$  k $\Omega$ ,  $R_3 = 2.5$  k $\Omega$ ,  $R_4 = 1$  k $\Omega$ ,  $R_5 = 4$  k $\Omega$ ,  $R_6 = 5$  k $\Omega$ , **solve for the values of the element voltages  $V_1$ ,  $V_2$ , and the node voltage  $u$ .** Show your work. You can use any circuit analysis techniques you have learned in this course.

**Solution:** We can solve this problem by NVA. Based on our labels in part (a), further label the current through  $R_2$ :



Write the KCL expression for the node with unknown voltage  $u$ :

$$I_1 + I_2 = 0.$$

Connect the node voltages with the element voltages:

$$V_1 = V_S - u,$$

$$V_2 = 0 - u = -u.$$

According to Ohm's Law, we know

$$I_1 = \frac{V_1}{R_1} = \frac{V_S - u}{R_1},$$

$$I_2 = \frac{V_2}{R_2} = \frac{-u}{R_2}.$$

Plug in the KCL expression:

$$\begin{aligned}\frac{V_S - u}{R_1} + \frac{-u}{R_2} &= 0 \\ \Rightarrow \frac{R_1 + R_2}{R_1 R_2} u &= \frac{V_S}{R_1} \\ \Rightarrow u &= \frac{R_2}{R_1 + R_2} V_S = \frac{4 \text{ k}\Omega}{1 \text{ k}\Omega + 4 \text{ k}\Omega} \times 5 \text{ V} = 4 \text{ V}.\end{aligned}$$

Therefore,

$$\begin{aligned}V_1 &= V_S - u = 5 \text{ V} - 4 \text{ V} = 1 \text{ V}, \\ V_2 &= -u = -4 \text{ V}.\end{aligned}$$

**Alternative solution:** We know  $R_1$  and  $R_2$  form a voltage divider for  $V_S$ . From the voltage divider equation we learned from the lecture, the node voltage

$$u = \frac{R_2}{R_1 + R_2} V_S = \frac{4 \text{ k}\Omega}{1 \text{ k}\Omega + 4 \text{ k}\Omega} \times 5 \text{ V} = 4 \text{ V}.$$

Therefore,

$$\begin{aligned}V_1 &= V_S - u = 5 \text{ V} - 4 \text{ V} = 1 \text{ V}, \\ V_2 &= -u = -4 \text{ V}.\end{aligned}$$

(e) (6 points) Given  $V_S = 5 \text{ V}$ ,  $R_1 = 1 \text{ k}\Omega$ ,  $R_2 = 4 \text{ k}\Omega$ ,  $R_3 = 2.5 \text{ k}\Omega$ ,  $R_4 = 1 \text{ k}\Omega$ ,  $R_5 = 4 \text{ k}\Omega$ ,  $R_6 = 5 \text{ k}\Omega$ .

- i. **If we combine  $R_3$ ,  $R_4$ ,  $R_5$ , and  $R_6$  as an equivalent resistor  $R_{\text{eq}}$  connecting between the nodes A and B, what is the value of  $R_{\text{eq}}$ ?**
- ii. **What are the values of the current  $I_3$  and the power dissipated by  $R_3$ ?** Show your work.

**Solution:** Between nodes C and B, we have  $R_4$  and  $R_5$  in series, and their equivalent resistor is in parallel with  $R_6$ , so the equivalent resistance for  $R_4$ ,  $R_5$ , and  $R_6$  between nodes C and B is

$$R_{\text{eq},C-B} = \frac{(R_4 + R_5)R_6}{(R_4 + R_5) + R_6} = \frac{(1 \text{ k}\Omega + 4 \text{ k}\Omega) \times 5 \text{ k}\Omega}{1 \text{ k}\Omega + 4 \text{ k}\Omega + 5 \text{ k}\Omega} = 2.5 \text{ k}\Omega.$$

$R_{\text{eq},C-B}$  is in series with  $R_3$ , so

$$R_{\text{eq}} = R_{\text{eq},C-B} + R_3 = 2.5 \text{ k}\Omega + 2.5 \text{ k}\Omega = 5 \text{ k}\Omega.$$

The current  $I_3$  is the same as the current through  $R_{\text{eq}}$  if we replace  $R_3$ ,  $R_4$ ,  $R_5$ , and  $R_6$  with  $R_{\text{eq}}$ , i.e.

$$I_3 = \frac{V_S}{R_{\text{eq}}} = \frac{5 \text{ V}}{5 \text{ k}\Omega} = 1 \text{ mA}.$$

Therefore, the power dissipated by  $R_3$  is

$$P_{R_3} = I_3^2 R_3 = (1 \text{ mA})^2 \times 2.5 \text{ k}\Omega = 2.5 \text{ mW}.$$

#### 4. Capacitive TouchSki (12 points)

One of your friendly lab TAs is preparing to go skiing for the first time! As excited as she is, she's very worried about losing her balance. To ease her mind, she decides to apply what she knows about capacitors to create a circuit that will indicate if there is excessive force applied to either ski.

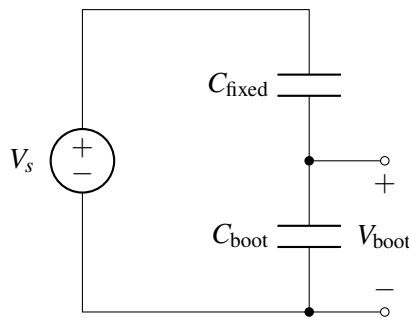
In this question, we will examine a force-sensing circuit for a single ski.

- (a) (3 points) To create a sufficiently large capacitance, your TA affixes conductive plates to both the bottom of the boot and the top of the ski, with a thin insulating layer with permittivity  $\epsilon$  in between. The boot has area  $A_{\text{boot}}$  and overlaps completely with the ski. Measurements show that the thickness  $t$  of the insulating layer varies with the force  $F$  as  $t = \frac{1}{kF}$ , where  $k$  is some constant. **Write the capacitance  $C_{\text{boot}}$  as a function of the force applied, the area of the boot, and constants.**

**Solution:**

$$C_{\text{boot}} = \epsilon \frac{A_{\text{boot}}}{t} = \epsilon A_{\text{boot}} k F$$

- (b) (4 points) In order to measure the capacitance  $C_{\text{boot}}$ , you are given the following circuit. Assume the capacitors have no initial charge before connected to the voltage source.

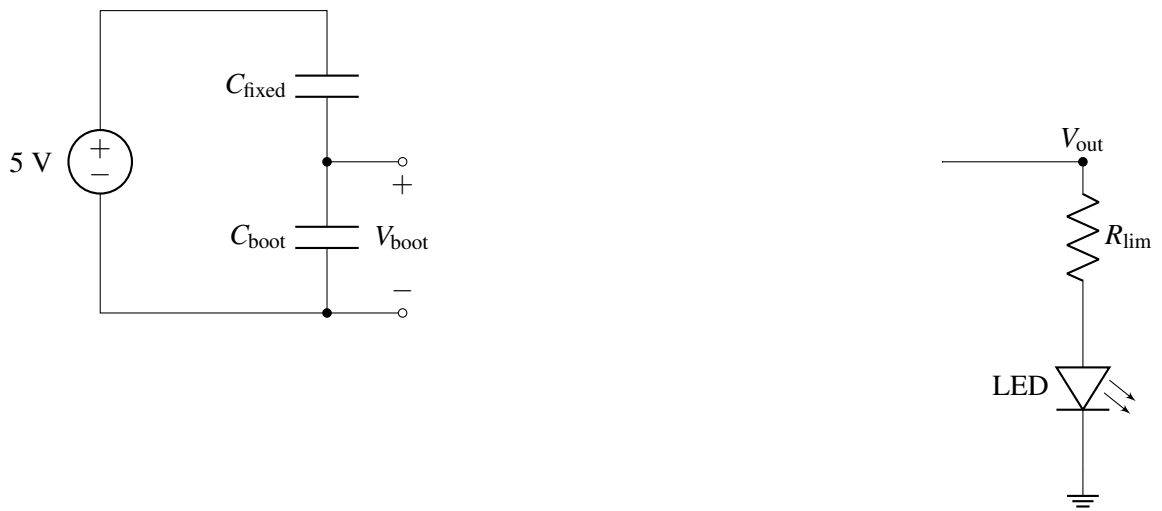


**What is the voltage  $V_{\text{boot}}$  in terms of  $V_s$ ,  $C_{\text{boot}}$ , and  $C_{\text{fixed}}$ ?**

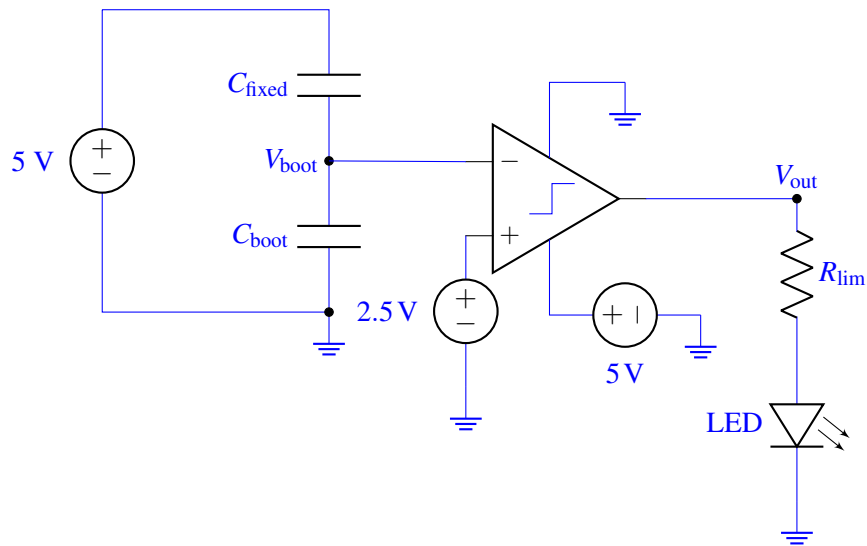
**Solution:**

$$V_{\text{boot}} = \frac{C_{\text{fixed}}}{C_{\text{boot}} + C_{\text{fixed}}} V_s$$

- (c) (5 points) Now, you'd like to control an LED based on the force applied to the ski. Redraw the following circuit in your answer sheet, **complete the circuit so it sets  $V_{\text{out}} = 5\text{ V}$  when  $V_{\text{boot}} < 2.5\text{ V}$ , and  $V_{\text{out}} = 0\text{ V}$  when  $V_{\text{boot}} > 2.5\text{ V}$**  (you don't need to consider the special case when  $V_{\text{boot}} = 2.5\text{ V}$ ). You may use one comparator and up to two additional voltage sources.



**Solution:**



Note that the node  $V_{\text{boot}}$  should be connected to the negative terminal of the comparator, and the other end of  $C_{\text{boot}}$  should be grounded. The power rails  $V_{DD} = 5 \text{ V}$  and  $V_{SS} = 0 \text{ V}$  should be specified in your answer, however we will grant full credit if you label the correct  $V_{DD}$  and  $V_{SS}$  values without drawing the 5 V voltage source for  $V_{DD}$ .

### 5. Resistive Temperature Sensor (17 points)

Oh no! Predictably, your lab TA gets hurt on the first day of her ski trip and is instructed to ice her injury regularly. However, she's finding that the ice packs are often too cold or too warm and needs a way to track their temperature.

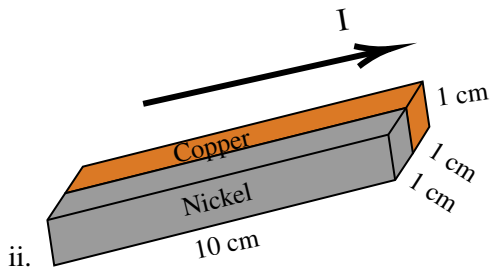
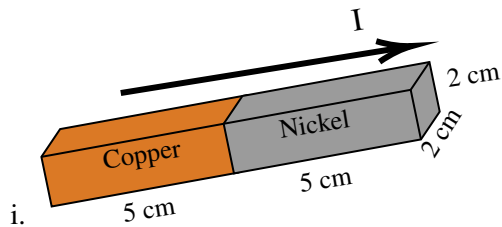
Fortunately, she remembers from 16A that the resistance of many resistors is dependent on temperature! Using this information, you decide to help her build a temperature-sensing device.

*Note: in this problem, temperature  $T$  is measured in the unit of Celsius.*

- (a) (6 points) You have different types of resistive bars available in your lab. **For each of the following two resistive bars, express the total resistance in terms of the given quantities and dimensions.**

$$\text{resistivity of copper} = \rho_{C_o}(1 + \alpha T) \Omega \cdot \text{cm}$$

$$\text{resistivity of nickel} = \rho_{N_o}(1 + \beta T) \Omega \cdot \text{cm}$$



#### Solution:

- i. For this bar,  $R_{Total} = R_{Copper} + R_{Nickel}$

$$R_{Copper} = \rho_C \frac{L}{A} = \frac{5}{4} \rho_{C_o} (1 + \alpha T)$$

$$R_{Nickel} = \rho_N \frac{L}{A} = \frac{5}{4} \rho_{N_o} (1 + \beta T)$$

$$R_{Total} = \frac{5}{4} (\rho_{C_o} (1 + \alpha T) + \rho_{N_o} (1 + \beta T))$$

- ii. For this bar,  $R_{Total} = \frac{R_{Copper} R_{Nickel}}{R_{Copper} + R_{Nickel}}$

$$R_{Copper} = \rho_C \frac{L}{A} = 10\rho_{C_o}(1 + \alpha T)$$

$$R_{Nickel} = \rho_N \frac{L}{A} = 10\rho_{N_o}(1 + \beta T)$$

$$R_{Total} = \frac{10(\rho_{C_o}(1 + \alpha T))(\rho_{N_o}(1 + \beta T))}{\rho_{C_o}(1 + \alpha T) + \rho_{N_o}(1 + \beta T)}$$



- (b) (5 points) You have the following circuit that has a temperature dependent resistive bar  $R_T$  and a resistor with fixed resistance  $R_f$ . **For this part only**, assume  $V_s = 5\text{ V}$ ,  $R_f = 4\text{ k}\Omega$ , and  $R_T$  has resistivity  $\rho_T = 100(1 + 0.01T)\ \Omega \cdot \text{cm}$  and cross-sectional area  $A = 1\text{ cm}^2$ .

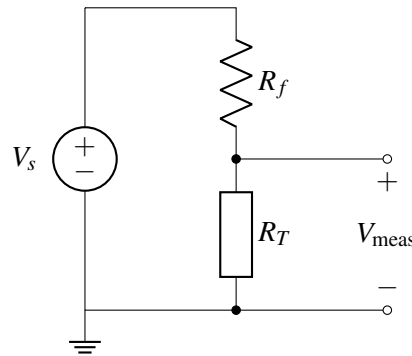


Figure 5.1: Circuit diagram for parts (b) and (c).

You want to be able to measure temperature  $T$  within the range  $-10^\circ\text{C} \leq T \leq 30^\circ\text{C}$ . You also want to limit the current flow through the resistive bar to be no more than 1 mA. **Find the minimum length of the resistive bar  $R_T$  such that the current limit is met for all temperatures in the specified range.**

**Solution:** Since  $V_s = 5\text{ V}$ , to have a maximum current of 1 mA, the minimum total resistance should be 5 k $\Omega$ .

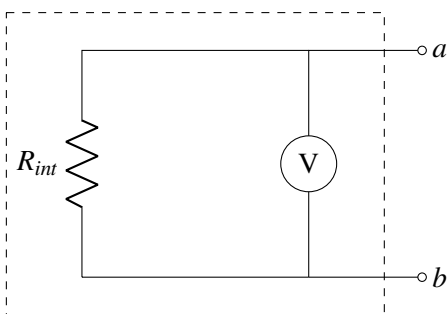
The total resistance in this circuit is  $R_f + \rho_T \frac{L}{A} = R_f + 100(1 + 0.01T) \frac{L}{A}$ , which increases with temperature, so the lowest total resistance occurs when the temperature is  $-10^\circ\text{C}$ . Therefore,

$$R_{Total} \geq R_f + 100(1 + 0.01 \times -10) \frac{L}{A} = 4\text{ k}\Omega + 90\ \Omega \cdot \text{cm} \frac{L}{1\text{ cm}^2} \geq 5\text{ k}\Omega$$

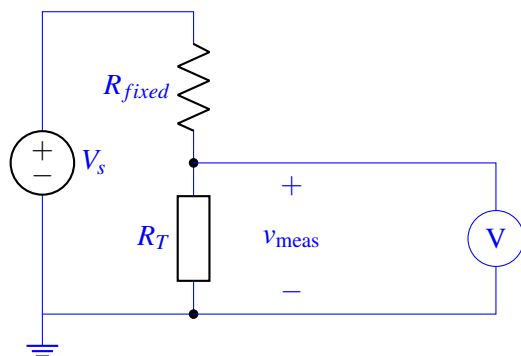
$$L \geq \frac{1000}{90}\text{ cm}$$

$$L_{min} = \frac{100}{9}\text{ cm} \approx 11.1\text{ cm}$$

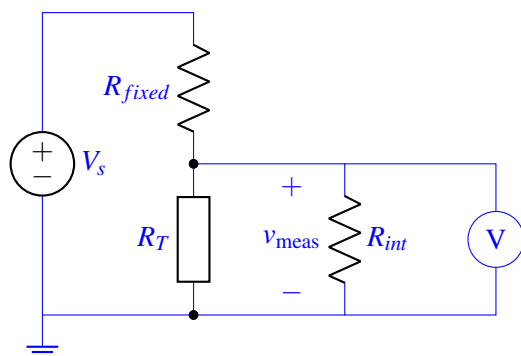
- (c) (6 points) Next, you are tasked with measuring the voltage across  $R_T$ .
- Draw how you would attach an ideal voltmeter to the circuit in Figure 5.1, in order to measure the voltage across  $R_T$ .**
  - Instead of an ideal voltmeter, you only have a practical voltmeter that can be modeled as an ideal voltmeter coming with a parallel internal resistance  $R_{int}$ , shown below. You connect the practical voltmeter to the same two nodes where you would attach the ideal voltmeter. Assuming  $R_T = 1000\ \Omega$  for this part, **determine the minimum value of  $R_{int}$  such that the equivalent resistance across the voltmeter is no less than 99% of  $R_T$ .**



**Solution:** Attach an ideal voltmeter:



For a practical voltmeter, the equivalent circuit is:



We find  $R_{eq}$  and set it to be equal to  $0.99R_T$ .

$$R_{eq} = \frac{R_T R_{int}}{R_T + R_{int}}$$

$$0.99R_T(R_T + R_{int}) = R_T R_{int}$$

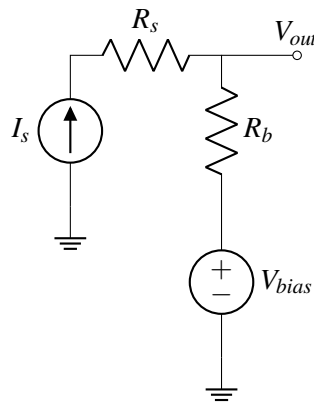
$$0.99(R_T + R_{int}) = R_{int}$$

$$R_{int} = \frac{0.99R_T}{0.01}$$

$$R_{int} = 99000 \, \Omega = 99 \, \text{k}\Omega$$

## 6. Data Conversion Circuits (22 points)

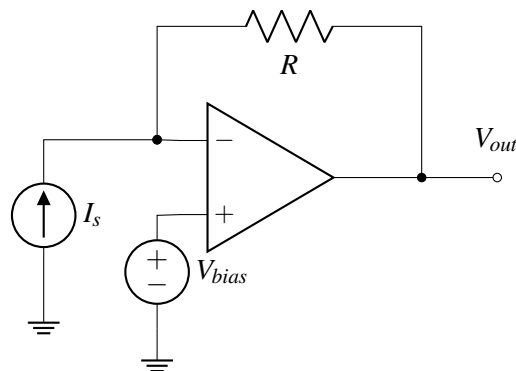
- (a) (4 points) Photonic circuits use light to communicate. We still need to convert the light into electricity by a photodiode to process it. We can model the photodiode as a current source  $I_s$ . Sometimes it is necessary at the receiver side to adjust the transmitted voltage level  $V_{out}$ , and one way to do this is using a voltage source  $V_{bias}$ . Consider this simple photonic receiver circuit:



Using *superposition*, solve for the voltage  $V_{out}$  in terms of  $I_s$ ,  $V_{bias}$ ,  $R_s$ , and  $R_b$ . Show your work.

**Solution:** First turn off the current source and turn on the voltage source. The circuit is an open circuit now, so  $V_{out,1} = V_{bias}$ . Then turn off the voltage source and turn on the current source, the current in the circuit is  $I_s$ , so  $V_{out,2} = I_s R_b$ . By superposition,  $V_{out} = V_{out,1} + V_{out,2} = V_{bias} + I_s R_b$ .

- (b) (4 points) The previous receiver circuit may have problems with loading. Instead, we may use an op-amp, such as in this circuit:

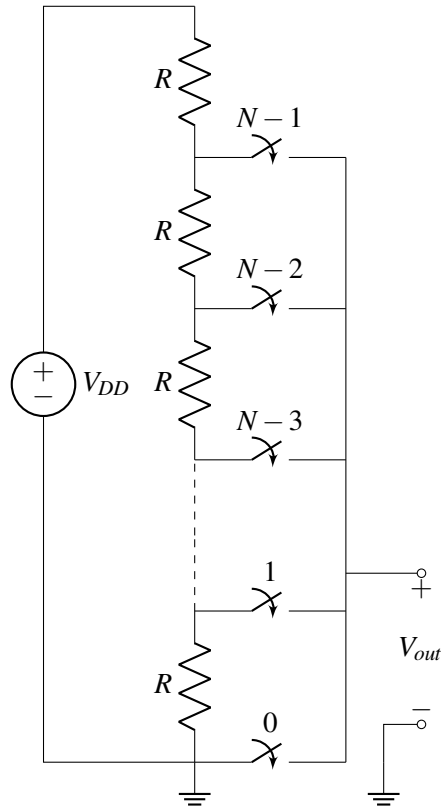


Calculate the voltage at the output  $V_{out}$  in terms of  $I_s$ ,  $V_{bias}$ , and  $R$ . Show your work. You will not receive full credit for directly copying a formula from your cheat sheet.

**Solution:** All the current  $I_s$  goes through the resistor, so the voltage drop on the resistor is  $V_R = -I_s R$ . From our golden rules, the voltage of the negative terminal of the op-amp must be  $V_{bias}$ . Therefore,

$$V_{out} = V_{bias} + V_R = V_{bias} - I_s R$$

- (c) (4 points) We need some circuits to convert between our analog voltage values and some digital representation stored in 1s and 0s. We mentioned digital-to-analog converter circuits, or DACs, in lecture. Let's inspect one here:



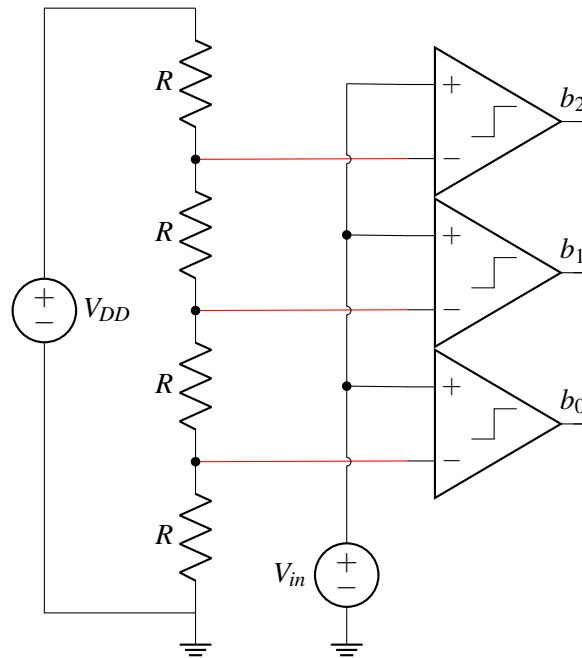
Note that there are  $N$  resistors and  $N$  switches in the circuit. Depending on some input digital code, one of the switches is closed, connecting the output to some node in the resistor ladder.

If only the  $i$ th switch is closed ( $0 \leq i \leq N - 1$ ), **what is the output voltage  $V_{out}$  in terms of  $V_{DD}$ ,  $i$ ,  $N$ , and  $R$ ?**

**Solution:** This is similar to our touchscreen problem. We can divide the resistor ladder into a top and bottom, then count up the number of resistors we have that form the top and bottom.

$$\begin{aligned}
 V_{out} &= \frac{R_{bot}}{R_{top} + R_{bot}} V_{DD} \\
 &= \frac{iR}{iR + (N - i)R} V_{DD} \\
 &= \frac{i}{N} V_{DD}
 \end{aligned}$$

- (d) (4 points) The dual to DAC circuits are analog-to-digital converters, or ADC circuits. Here is an example of one, using resistors and comparators:



*Note: The red wires in the diagram are regular wires, but have been colored to show that they do not touch the crossing black wires.*

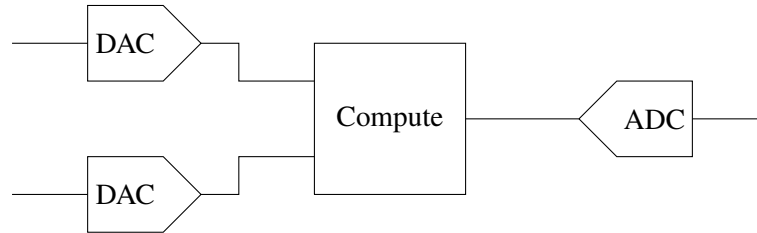
The resistor ladder gives us a set of reference voltages to compare against. We use a set of comparators to compare the input voltage  $V_{in}$  against these reference levels, and we get out a corresponding digital code  $b_0$ ,  $b_1$ , and  $b_2$ .

Assume that  $V_{DD} = 1\text{ V}$ , and that the comparators are connected to rails  $V_{DD} = 1\text{ V}$  and  $V_{SS} = 0\text{ V}$ . If  $V_{in}$  is  $0.3\text{ V}$ , **what are the outputs  $b_0$ ,  $b_1$ , and  $b_2$ ?**

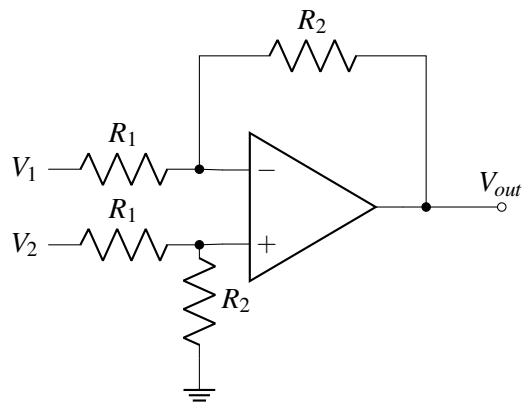
**Solution:** The reference voltages for the comparators, from bottom to top, are  $\frac{1}{4}V_{DD} = 0.25\text{ V}$ ,  $\frac{1}{2}V_{DD} = 0.5\text{ V}$ , and  $\frac{3}{4}V_{DD} = 0.75\text{ V}$ . Each comparator checks if  $V_{in}$  is greater than the reference voltage, outputting  $V_{DD}$  if it is greater and  $V_{SS}$  if not. We are given  $V_{in} = 0.3\text{ V}$ , so  $b_0 = V_{DD} = 1\text{ V}$ ,  $b_1 = V_{SS} = 0\text{ V}$ ,  $b_2 = V_{SS} = 0\text{ V}$ .

If you left your answer in terms of 1s and 0s, i.e.  $b_0 = 1, b_1 = 0, b_2 = 0$ , we will accept that for full credit.

- (e) (6 points) These DAC and ADC circuits help us represent numbers using voltage values. We would like to build some blocks that let us compute, e.g. add and multiply, with these numbers. We call this "analog computing," and we saw an example of this previously in the "artificial neuron" circuit. These analog compute circuits have potentially massive speed benefits over comparable digital compute circuits. (Take EECS151 and EE140 for more details).



Consider one such compute circuit below, the differential amplifier. This is a common circuit in audio amplifiers but is also a useful tool for mathematical computing. **Find the output  $V_{out}$  in terms of  $V_1$ ,  $V_2$ ,  $R_1$ , and  $R_2$ .**



**Solution:** Let's approach this problem with superposition. First we turn off  $V_2$  and keep  $V_1$  on. Note that we have no current flowing through the resistors at the positive terminal, so we effectively have ground at that node. Thus, this is just an inverting amplifier:

$$V_{out} = -\frac{R_2}{R_1}V_1$$

Next, we turn off  $V_1$  and keep  $V_2$  on. We have two things happening in this circuit. The first pair  $R_1$  and  $R_2$  connecting  $V_2$  to the positive terminal form a resistive voltage divider. Then the op-amp and the feedback  $R_1$  and  $R_2$  form a non-inverting amplifier:

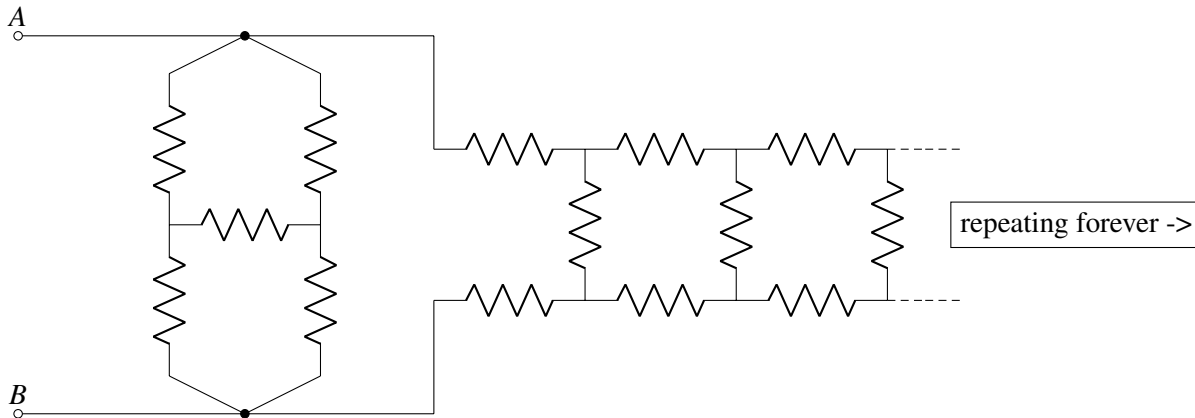
$$V_{out} = \frac{R_2}{R_1 + R_2} \frac{R_1 + R_2}{R_1} V_2 = \frac{R_2}{R_1} V_2.$$

Adding these together, we get:

$$V_{out} = -\frac{R_2}{R_1}V_1 + \frac{R_2}{R_1}V_2 = \frac{R_2}{R_1}(V_2 - V_1)$$

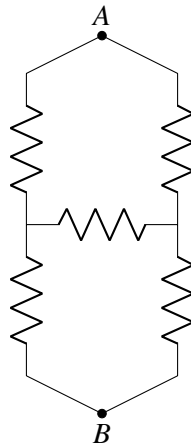
### 7. Aid to the Resistance (13 points)

The main ship of the Resistance Fleet is in trouble! They have recruited you to help fix the issue. The on-board technicians have determined that the resistor grid in the main console is faulty (one of the resistors must be fried). It is your job to replace the grid with something of equivalent resistance. However, because of severe budget cuts in the Resistance's EE department, **you can only use a single resistor connected between nodes  $A$  and  $B$  to replace the resistor grid**. The technicians hand you the diagram below of what the resistor grid looked like. **All resistors in the diagram have resistance value  $R$ .**

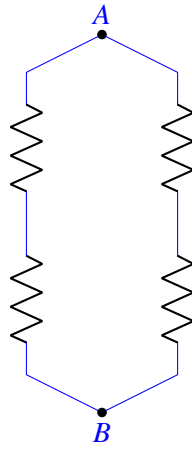


- (a) (5 points) **Find the equivalent resistance** of the following piece of the resistor grid between nodes  $A$  and  $B$  in terms of  $R$ .

*Hint: If a resistor has no current flowing through it, what is it equivalent to?*



**Solution:** By symmetry, we can argue that the voltage across the horizontal resistor is 0, i.e. there is no current flowing through the horizontal resistor (recall our resistor grid model for the resistive touch screen). Because of this, we can model the horizontal resistor as an open circuit without changing the behavior of the resistor grid, giving us the equivalent circuit below:



The equivalent resistance of this portion is:

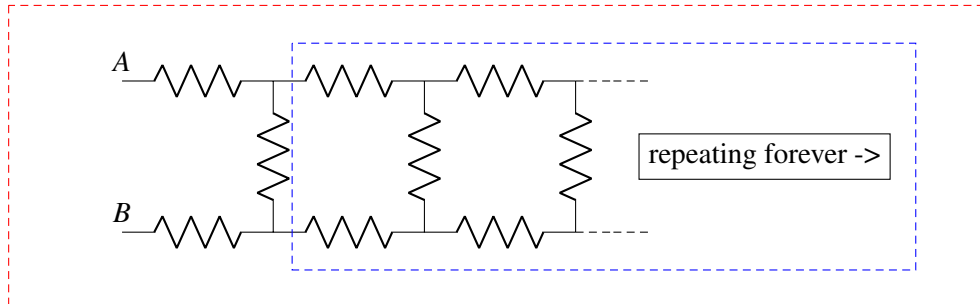
$$R_{eq} = \frac{(2R)(2R)}{2R + 2R} = R.$$



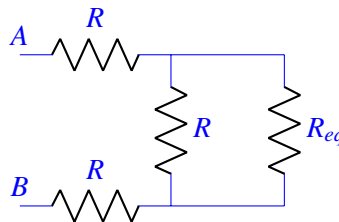
- (b) (5 points) **Find the equivalent resistance** of the following piece of the resistor grid between nodes  $A$  and  $B$  in terms of  $R$ .

*Hint: (i) Because the pattern is infinite, the equivalent resistance of the circuit in the red box and the equivalent resistance of the circuit in the blue box are equal.*

*(ii) The solutions to the quadratic equation  $ax^2 + bx + c = 0$  are  $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .*



**Solution:** The hint points us to the below equivalent circuit:



We can now set up an equation for  $R_{eq}$  in terms of itself:

$$\begin{aligned}
 R_{eq} &= R + R \parallel (R_{eq} + R) \\
 R_{eq} &= 2R + \frac{RR_{eq}}{R + R_{eq}} \\
 R_{eq} - 2R &= \frac{RR_{eq}}{R + R_{eq}} \\
 RR_{eq} &= (R_{eq} - 2R)(R + R_{eq}) \\
 0 &= R_{eq}^2 - 2RR_{eq} - 2R^2 \\
 R_{eq} &= \frac{2R \pm \sqrt{4R^2 + 8R^2}}{2} \\
 R_{eq} &= \frac{2R \pm \sqrt{12R^2}}{2} \\
 R_{eq} &= R \pm \sqrt{3}R \\
 R_{eq} &= (1 + \sqrt{3})R
 \end{aligned}$$

Notice that we took the positive solution because resistance cannot be negative.

- (c) (3 points) Suppose the equivalent resistance for the piece of resistor grid in part (a) is  $\alpha R$ , and the equivalent resistance for the piece of resistor grid in part (b) is  $\beta R$ , where  $\alpha$  and  $\beta$  are known real

numbers for this part. **What should be the value of the resistor you use to replace the entire grid with, in terms of  $R$ ,  $\alpha$ , and  $\beta$ ?**

**Solution:** The pieces we see above are put in parallel in the final resistor grid, making the equivalent resistance of the entire grid:

$$R_f = \frac{R_{eq_a} R_{eq_b}}{R_{eq_a} + R_{eq_b}} = \frac{\alpha\beta}{\alpha + \beta} R$$

In order to replace the resistor grid, we would need to purchase a resistor of resistance  $R_f$ , as specified above.

## 8. Explosive Tesla Coil!! (19 points)

Renowned as the father of alternating current technology, the brilliant Nikola Tesla created countless inventions which were truly beyond his time. One of such inventions was the Tesla Coil, a device that could continuously generate safe lightning! While we do not yet have all the background knowledge needed for the full circuitry, we can still make an effective model of this device with a capacitor network.

Loosely stated, the Tesla coil circuit charges up a specially-designed, massive capacitor (labeled  $C_T$  here) until it reaches a certain threshold of charge. The effective capacitor model is shown in Figure 8.1, where  $V_S$  is the charging voltage source, each  $C$  are identical charge-loading capacitors, and  $C_T$  is the capacitor that models our Tesla Coil.

The capacitor  $C_T$  charges over a repeated series of cycles. Each cycle involves two stages:

- Stage A – The  $\phi_B$  switches in Figure 8.1 open and then the  $\phi_A$  switches close. In this stage the loading capacitors  $C$  are charged by  $V_S$ .
- Stage B – The  $\phi_A$  switches in Figure 8.1 open and then the  $\phi_B$  switches close. In this stage the loading capacitor  $C$  charges are shared with the Tesla coil capacitor  $C_T$ .

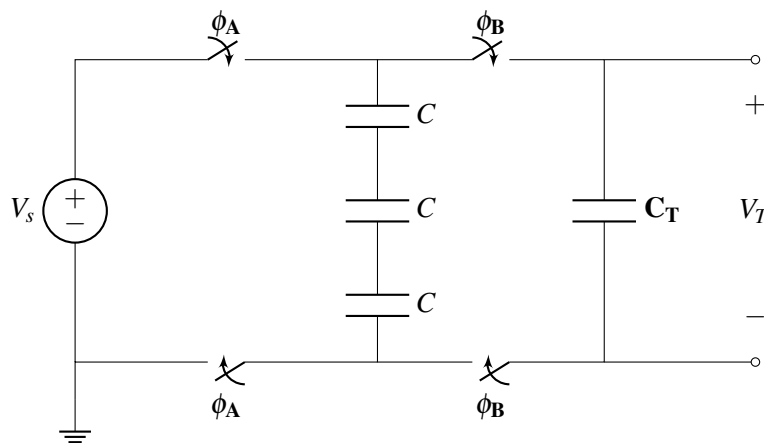
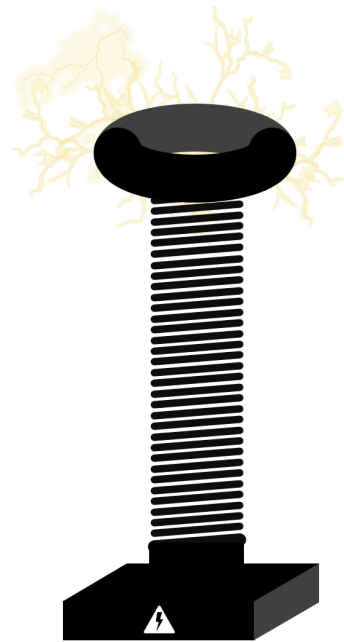


Figure 8.1: Tesla coil effective circuit model.

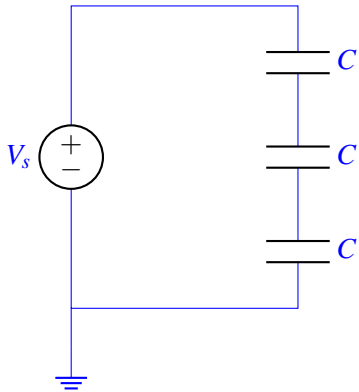


- (a) (4 points) **Draw the equivalent circuit diagram for each stage (A and B).** To receive full credit, make sure each diagram you draw has only the relevant circuit elements, i.e. the diagram should not include any elements that are not connected in a closed circuit.

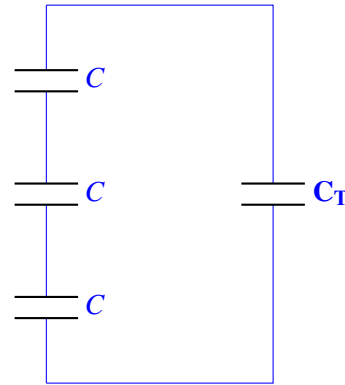
**Solution:**

The following circuit diagrams are drawn:

For stage A:



For stage B:



- (b) (3 points) If we would like to model the three identical loading capacitors as a single equivalent capacitor labeled  $C_L$ , **what value should we choose for  $C_L$  to ensure that the circuit has the same behavior?** Your answer should be in terms of  $C$ .

**Solution:**

One can use the equivalent series equation twice to determine this value:

$$C_L = \left( \left( \left( \frac{1}{C} + \frac{1}{C} \right)^{-1} \right)^{-1} + \frac{1}{C} \right)^{-1} = \left( \frac{1}{C} + \frac{1}{C} + \frac{1}{C} \right)^{-1} = \left( \frac{3}{C} \right)^{-1} = \frac{C}{3}$$

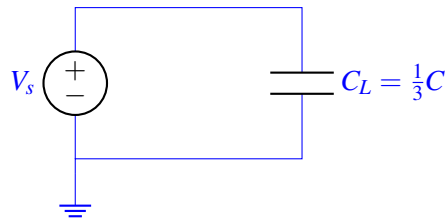
- (c) (6 points) Suppose that all capacitors in the device start off without any charge. **Compute the charge  $Q_T$  and the voltage  $V_T$**  across the Tesla coil capacitor  $C_T$  after the circuit completes a cycle (going from stage A to stage B).

All final solutions should be in terms of the known circuit constants ( $V_s$ ,  $C$ , and  $C_T$ ).

*Hint: You may use the simplified equivalent loading capacitor model  $C_L$  through your work, then plug  $C$  back in at the end.*

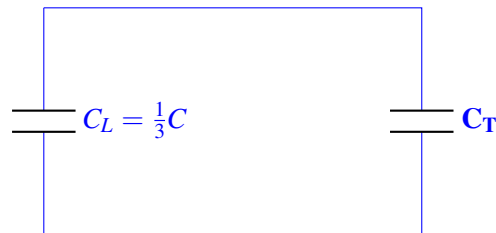
**Solution:**

From our setup in (a), we first determine the charge  $Q_L^{(A)}$  on the loading capacitor  $C_L$  after stage A:



$$Q_L^{(A)} = C_L V_s = \frac{1}{3} C V_s$$

Next we transition to stage *B* and recognize that the total charge on the upper node is  $Q_L^{(A)}$ . From here we can identify the voltage  $V_T$  by assembling the equivalent capacitance that this charge resides on, which is just  $C_L$  and  $C_T$  in parallel.



$$C_{eq} = C_L + C_T$$

Lastly we determine  $V_T$ , and consequently  $Q_T$ :

$$V_T = \frac{Q_L^{(A)}}{C_{eq}} = V_s \left( \frac{C}{C + 3C_T} \right)$$

$$Q_T = C_T V_s \left( \frac{C}{C + 3C_T} \right)$$

(d) (6 points) Now suppose that the Tesla coil capacitor  $C_T$  starts with an initial charge  $Q_0$ , which was collected as a result of previous cycles.

- i. **Compute the charge  $Q_T$  and the voltage  $V_T$  across the Tesla coil capacitor  $C_T$  after the circuit completes a cycle (going from stage *A* to stage *B*).** For simplicity, you can assume the loading capacitors still start off without any charge.
- ii. **Compute the ratio  $\sigma$  of energy stored on  $C_T$  before and after this cycle** (so  $\sigma = E_{\text{after}}/E_{\text{before}}$ ). You can get partial credit for writing the expressions for  $E_{\text{before}}$  and  $E_{\text{after}}$ .  
**At which value of initial charge  $Q_0$  do we no longer add energy to the coil after a cycle?**

All final solutions should be in terms of the known circuit constants ( $V_s$ ,  $C$ ,  $C_T$ , and  $Q_0$ ).

*Hint: It may be helpful to simplify the final answer of  $\sigma$  in the form:*

$$\sigma = \left( \frac{3 \left( \frac{C_T}{C} \right) + \_\_\_}{3 \left( \frac{C_T}{C} \right) + \_\_\_} \right)^2$$

where the  $\_\_\_$  spaces are yet to be discovered by you!

**Solution:**

The work in computing  $Q_L^{(A)}$  from the previous subpart still holds. In fact, the only change in this new scenario is that during  $\phi_B$  the total charge being shared across  $C_L$  and  $C_T$  in parallel is now  $Q_L^{(A)} \rightarrow Q_L^{(A)} + Q_0$ . Thus we end up with:

$$V_T = \frac{Q_L^{(A)} + Q_0}{C_{eq}} = \frac{C V_s + 3Q_0}{C + 3C_T} = V_s \left( \frac{C + (3Q_0/V_s)}{C + 3C_T} \right)$$

$$Q_T = C_T V_s \left( \frac{C + (3Q_0/V_s)}{C + 3C_T} \right)$$

As expected, we should attain the same results as in the previous subpart for the special case that  $Q_0 = 0$ . Now we turn to compute the change in stored energy on  $C_T$  using the formula  $E_{\text{stored}} = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$ . In this context, our ratio becomes  $\frac{E_{\text{after}}}{E_{\text{before}}} = \left( \frac{Q_{\text{after}}}{Q_{\text{before}}} \right)^2$ , thus

$$\sigma = \frac{E_{\text{after}}}{E_{\text{before}}} = \left( \frac{C_T V_s \frac{C + (3Q_0/V_s)}{C + 3C_T}}{Q_0} \right)^2 = \left( \frac{3 \left( \frac{C_T}{C} \right) + \frac{C_T V_s}{Q_0}}{3 \left( \frac{C_T}{C} \right) + 1} \right)^2$$

We are increasing the energy stored in the coil  $C_T$  whenever  $\sigma > 1$ . Thus at the point  $\sigma = 1$  is when we are maximally charged and do not increase the capacitor energy. From our expression, this occurs when  $Q_0 = C_T V_s$ . This is when the coil is charged to the supply voltage level  $V_T = V_s$ .