EECS 16A — Designing Information Devices and Systems I

Fall 2015 — Anant Sahai, Ali Niknejad

Final Exam

Exam location: You're Done!!

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________

(last) (first) (signature)

PRINT your Unix account login: ee16a-____________

PRINT your discussion section and GSI (the one you attend): ________________________________

Row Number (front row is 1): ________ Seat Number (left most is 1): ________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

Section 0: Pre-exam questions (3 points)

1. **What was your favorite lab/homework-problem in 16A? What did you like best about it? (1 pt)**

   Solution: I liked the "Removing the Troll," Justin Bieber, and Pikachu problems.

2. **Describe how it makes you feel when you work with a TA or friend to understand something? (2 pts)**

   Solution: It feels great! Teamwork and challenge, hand in hand.

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 1: Straightforward questions (50 points)

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section. **You get two drops: do 5 out of the following 7 questions. (We will grade all 7 and keep the best 5 scores.)** *Each problem is worth 10 points. No bonus for getting them all right so skip anything that is taking too much time.*

3. Finding determinant

Compute the determinant of this 3×3 matrix by using Gaussian elimination.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & 4 & 5 \end{bmatrix}$$

Solution: We do Gaussian elimination on A

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

At this point, we can see that the determinant is 2 as it is the product of the diagonal elements of a triangular matrix (because it is equal to the volume of high-dimensional parallelogram defined by the columns of the matrix).

4. Back to Basis

Find the matrix that changes the coordinate representation of a vector in the basis $\{\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ to a coordinate representation in the basis $\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix}\}$. No need to simplify.

Solution: Let us define

$$A = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 4 & -3 \end{bmatrix}.$$

The matrix that transforms the coordinates from the basis given by matrix A to the basis given by matrix B is

$$T = B^{-1}A.$$

If one chooses to do the simplications, it is seen that

$$T = \frac{1}{9}\begin{bmatrix} 15 & 6 \\ 11 & 5 \end{bmatrix}.$$

There are also different ways of coming up with this transformation.

5. Eigenspaces

Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 3$ for the following matrix A:

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

Solution:

For $\lambda = 3$ a basis for the eigenspace is given by the vectors $\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \}$.

This can be seen by looking at $A - 3I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$. Doing Gaussian elimination on $A - 3I$, we get $\begin{bmatrix} -4 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see that the last two columns are identical and the first is -2 times either of them. This immediately gives the nullspace and hence the eigenspace of the original matrix. It is a two-dimensional eigenspace hence the basis has two vectors.

6. Show It

Let n be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of k linearly dependent vectors in $\mathbb{R}^n$. Show that for any $(n \times n)$ matrix A, the set $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ is a set of linearly dependent vectors.

Your argument should be concise and mathematical.

Solution: (This question is exactly from Midterm 1, as you may have noticed.)

Since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are linearly dependent, by definition there exist coefficients $\{\alpha_i\}$ (not all zero) such that

$$\sum_{i=1}^{k} \alpha_i \vec{v}_i = \vec{0}.$$

Thus, left-multiplying by A:

$$A\sum_{i=1}^{k}\alpha_i\vec{v}_i = A\vec{0} = \vec{0}$$

And so:

$$\sum_{i=1}^{k}\alpha_i(A\vec{v}_i) = \vec{0}$$

for coefficients $\{\alpha_i\}$ (not all zero). Therefore, by definition, the set of vectors $\{A\vec{v}_i\}$ are linearly dependent.

7. Linear Recurrence

Suppose the sequence $x_0, x_1, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots$ is defined recursively as follows:

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 1 \\ x_t &= -3x_{t-1} + 4x_{t-2} \end{aligned}$$

So, the first few terms in this sequence are: $0, 1, -3, 13, -51, \ldots$.

It turns out that by using matrix notation and diagonalization, we can get:

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = PD^{t-1}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where D is a diagonal matrix. **Find P and D.** (You don't need to invert P.)

Solution: The recurrence relation is defined by the following matrix $A = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}$ and the formula $\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = A^{t-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, we just need to diagonalize A.

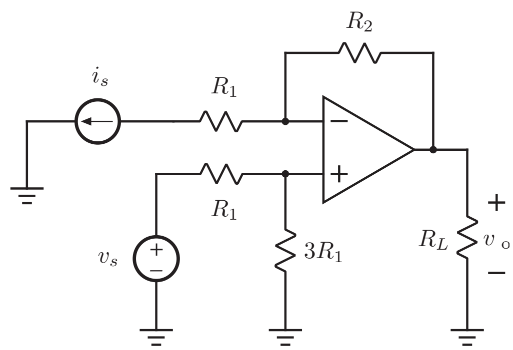
To find the eigenvalues $\lambda = -4$ and $\lambda = 1$, we need to find the roots of the characteristic polynomial given by $det(A - \lambda I) = 0$. Using these eigenvalues we can then find the corresponding eigenvectors $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The columns of P contain the eigenvectors and the diagonal entries of D contain the corresponding eigenvalues:

$$P = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$$

8. Golden Rules

Find v_o, the voltage at the output of the opamp (i.e. across the load R_L) for general resistor and source values.



Solution: Using KCL at the v_- node we find that $i_s = \frac{v_0 - v_-}{R_2}$ because, by the golden rules, no current flows into the opamp. Using the golden rules again we know that $v_- = v_+$. The voltage at $v_+ = \frac{3R_1}{R_1+3R_1}V_s = \frac{3}{4}V_s$ which is the result of a voltage divider. Putting it all together we get:

$$i_s = \frac{v_0 - \frac{3}{4}V_s}{R_2}$$

$$i_s R_2 = v_0 - \frac{3}{4}V_s$$

$$v_0 = i_s R_2 + \frac{3}{4}V_s$$

An alternative solution uses superposition. Turning off the current source makes it an open circuit. By the Golden rules, specifically the fact that no current flows through negative terminal of the op-amp, then no current flows through R_2 and output is simply the same as v^-

$$v_o^{(v_s)} = v^- = v^+ = \frac{3}{4}v_s \tag{2}$$

The last step is from the same voltage divider as above. Now turn off v_s, which shorts R_1 in parallel with $3R_1$. Since no current flows into the positive terminal of the op-amp, these resistors effectively ground v^+, which makes v^- also zero. This implies that

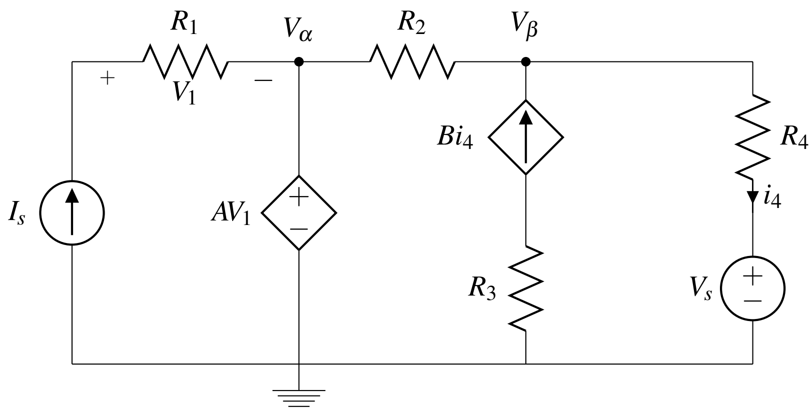
$$v_o^{(i_s)} = v^- + i_s R_2 = i_s R_2 \tag{3}$$

The complete response is given by the superposition

$$v_o = v_o^{(v_s)} + v_o^{(i_s)} = \frac{3}{4}v_s + i_s R_2 \tag{4}$$

9. Nodal Analysis

For the following circuit, which of the (a),(b),(c),(d) set of equations is correct (one of them is definitely correct) and sufficient to give us both V_α, V_β assuming that we knew I_s, V_s and all the values for R_1, A, R_2, B, R_3, R_4?



(a)	(b)	(c)	(d)
$V_\alpha = AI_sR_1$	$V_\alpha = AI_sR_1$	$V_\alpha = AI_sR_1$	$V_\alpha = AI_sR_1$
$V_\beta = \frac{(B+1)V_sR_2 + AI_sR_1R_4}{(B+1)R_2+R_4}$	$V_\beta = -100V$	$V_\beta = \frac{(B-1)V_sR_2 - AI_sR_1R_4}{(B-1)R_2-R_4}$	$V_\beta = \frac{(B+1)V_sR_2 - AI_sR_1R_4}{(B+1)R_2-R_4}$

Solution: Because current flowing through resistor R_1 is I_s, by Ohm's law we have $V_1 = I_sR_1$. Then looking at the dependent source, we see $V_\alpha = AV_1 = AI_sR_1$.

Then doing nodal analysis at V_β we have

$$\frac{V_\beta - V_\alpha}{R_2} + \frac{V_\beta - V_s}{R_4} = Bi_4.$$

By Ohm's law, we see that $i_4 = \frac{V_\beta - V_s}{R_4}$. Substituting i_4 in the equation above and simplifying we get

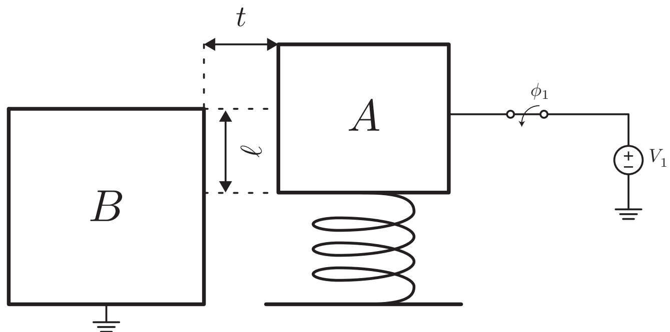
$$V_\beta = \frac{(B-1)V_sR_2 - V_\alpha}{(B-1)R_2 - R_4}.$$

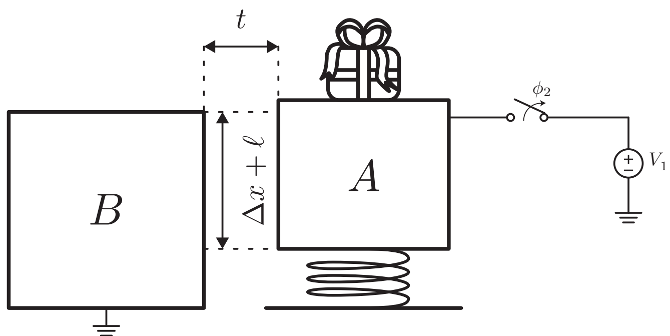
Because $V_\alpha = AI_sR_1$ we see that the solution is C.

Section 2: Free-form Problems (101+15 points)

10. Airport (41+10 points)

In this question you will design a scale (illustrated below) for an airport check-in counter. A bag is placed on the metal platform A and will move the platform down with respect to the fixed plate B on the left. The vertical displacement, Δx, is proportional to the weight of the bag. A parallel plate capacitor is formed by the metal platform A and the fixed metal plate B. The fixed metal plate B is connected to ground. Assume the separation t and the width w (into the page) stay constant along the entire platform and plate. (You do not need the specific value for t or w to solve any part of this problem.) When there is no bag on the platform, let's call the capacitance C_{min}.





(a) (5 points) In the first phase ϕ_1, *before placing the bag on the platform*, the platform is charged to a voltage $V_1 = 5V$. **Calculate the amount of charge Q_1 on the capacitor in terms of C_{min}.** Just tell us the charge on the platform A.

Solution:

The charge on a capacitor is given by the equation:

$$Q = CV$$

The capacitance in this equation is C_{min} and the voltage is 5V.

$$Q = 5C_{min} \text{ Coulombs}$$

(b) (8 points) **Calculate the capacitance between the platform and the plate when the platform is displaced by a positive distance** Δx. This displacement makes the effective plates of the capacitor bigger. Assume the bag is non-conductive plastic. Assume parallel plate capacitance (ignore fringing fields). Express your answer in terms of only C_{min}, ℓ, and Δx.

Solution:

$$C = C_{min} + C_{min}\frac{\Delta x}{\ell}$$
$$C = C_{min}\left(\frac{\Delta x + \ell}{\ell}\right)$$

Alternatively...

$$C_{min} = \frac{\varepsilon w \ell}{t}$$
$$C = \frac{\varepsilon w(\ell + \Delta x)}{t}$$
$$C = C_{min} + C_{min}\frac{\Delta x}{\ell}$$
$$C = C_{min}\left(\frac{\Delta x + \ell}{\ell}\right)$$

(c) (5 points) In the second phase ϕ_2, the switch is opened (disconnecting the voltage source) and then the bag is placed on the platform A. **Calculate the second phase voltage V_2 across the capacitor in terms of only Δx, ℓ, and the first phase voltage $V_1 = 5V$.**

(Hint: As you increase Δx, the voltage V_2 should decrease.)

Solution:

$$Q = 5C_{min}$$

From part (a)

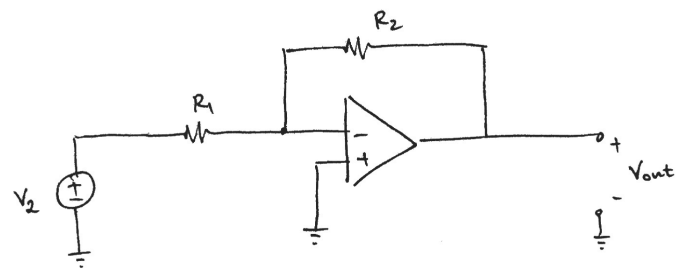
$$C = C_{min}\left(\frac{\Delta x + \ell}{\ell}\right)$$

From part (b)

$$V_2 = Q/C = \frac{5\ell}{\Delta x + \ell}$$

(d) (10 points) We decide that we want to amplify the voltage on the capacitor by a factor of -2. Assuming (for now) that the capacitor acts like an ideal voltage source whose value is V_2 Volts (as shown below), **please draw a circuit that could perform this task.** You are allowed to use wires, resistors, and a single golden-rule op-amp. Your circuit should conceptually fit in the box below.

Solution:



$$\frac{R_2}{R_1} = 2$$

Any values for R_2 and R_1 will work as long as they satisfy that ratio. For example, $R_1 = 1k\Omega$ and $R_2 = 2k\Omega$.

YOUR CIRCUIT GOES HERE!!

(e) (5 points) Now, instead of an ideal voltage source, we want to connect the circuit from the previous part to the capacitor in the bag weighing setup. It turns out that the circuit in the previous part would load (i.e. draw current to/from) the capacitor if we attached it. **Explain why this happens.**

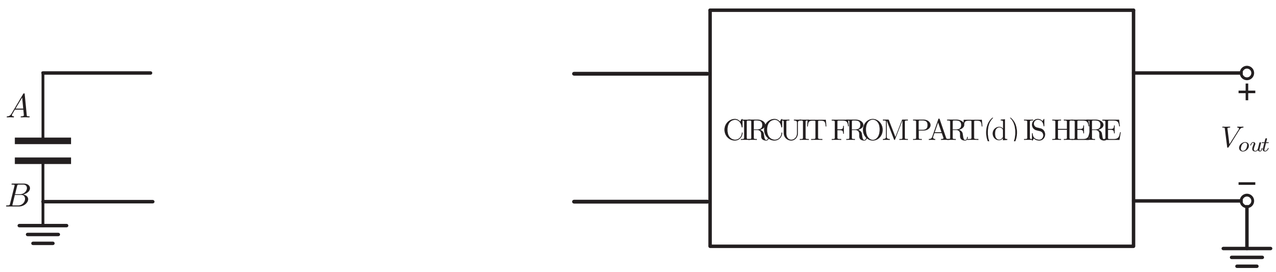
(Hint: explain what is the equivalent resistance that the capacitor sees to ground when connected to the circuit in the previous part. This might require a different approach than the black-box approach to calculating Thevenin equivalence.)

Solution:

$R_{in} = R_1$. This is because the $-$ terminal of the opamp is a virtual ground, since the $+$ terminal is grounded and the opamp is connected in negative feedback. So, the effective resistance to ground is just R_1. Since there is an input resistance presented across the terminals of the capacitor, there is a path for the capacitor to discharge in steady state. This path will draw current until the capacitor is discharged.

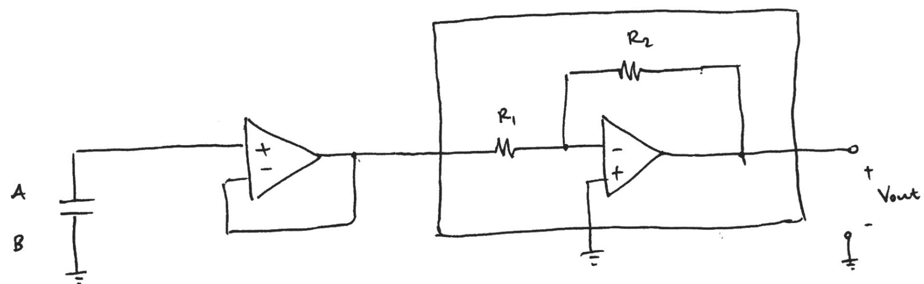
(f) (8 points) **Draw a circuit that you could place between the capacitor and the circuit from part (d) that would prevent the effects of the loading.** You are allowed to use wires, resistors and one golden-rule op-amp for your new circuit. You don't have to redraw the circuit from part (d). The box below is just to remind you where it would go.

NEW CIRCUIT GOES HERE!!



Solution:

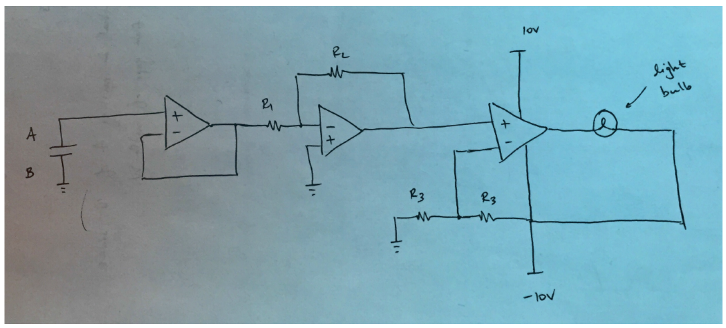
Use a buffer



(g) (BONUS 10 points) One last thing. We would like to turn on a light bulb if the bag were greater than 50 lbs. When $\Delta x = \ell$, the weight of the bag is exactly 50 lbs.

Attached to the output of your previous circuits (the whole setup drawn in (f)), **please draw a new circuit that turns a light on if $\Delta x > \ell$ and turns the light off if $\Delta x < \ell$.** Assume the light bulb acts like a 100Ω resistance and it turns on if a nonzero current flows through it. In addition to the lightbulb, you are allowed to use wires, resistors, and a single op-amp in your design for this part. The supply voltages available to you are $10V$ and $-10V$ and you can assume that the op-amp is connected to these as its supply rails. You don't have to redraw what was in (f) – you can start with V_{out} being available to you.

Solution:



The R_3, R_3 here set up a voltage divider that puts $-5V$ at the negative terminal of the opamp being used as a comparator. A value like $R_3 = 2k\Omega$ can be used. The reason we compare to $-5V$ is that we have a gain of -2 and when the weight is 50lbs, the capacitance has increased by a factor of 2. So the input to the buffer in that case would be $2.5V$ which when gained by -2 would give $-5V$. If the weight were more, then the voltage on the capacitor would be lower, which would make the inverted voltage entering the comparator higher than $-5V$. This would put 10V on the light-bulb and current will flow, turning the bulb on. If the weight were less, then the voltage on the capacitor would be higher and after inverting, the voltage entering the comparator would be lower than $-5V$. The comparator would then put -10V on the lightbulb and there would be no voltage drop across the light-bulb. Hence it would be off.

11. Channel Equalization (33 points)

In HW14 and lecture, we have talked about transmitting information through a wireless channel with echos. You don't have to remember the HW problem to solve this question, all relevant information is self-contained here.

If we periodically transmit an n-long vector $\vec{x}$ through a channel with circulant echo matrix H, we receive the n-long vector $\vec{y} = H\vec{x}$.

Helpful Background: In OFDM, we transmit information on the eigenvectors of H. We let

$$\vec{s} = \sum_{\ell=1}^{n} \alpha_\ell \vec{u}_\ell$$

where $\{\alpha_\ell\}$ is the set of messages we are sending and $\vec{u}_\ell$ are eigenvectors of H. If we then choose $\vec{x} = \vec{s}$ itself, then we will receive the n-long vector

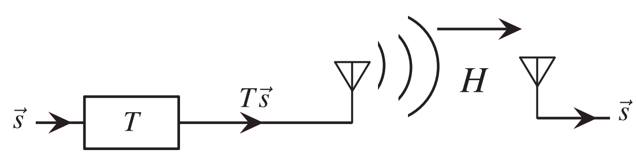
$$\vec{y} = H\vec{x} = \sum_{\ell=1}^{n} \lambda_\ell \alpha_\ell \vec{u}_\ell$$

where $\{\lambda_\ell\}$ is the sequence of eigenvalues of H. Then, if the receiver knows all the λ_ℓ, it can solve for α_ℓ, as long as none of the λ_ℓ are zero.
By changing coordinates to the orthonormal eigenbasis U (consisting of the $\vec{u}_\ell$), we can write

$$U^*\vec{y} = \Lambda\vec{\alpha}$$

where Λ is a diagonal matrix and $U^*\vec{y}$ is the received vector after a change of basis.
One of your homework problems mentioned how pilot tones can be used to estimate Λ. However, a very simple device — for example, a wireless decoder in your clothes — might not be powerful enough to estimate H and divide by the λ_ℓ's. Our goal is to make our device as simple as possible, so to do that we'll make the transmitter (which is plugged into the wall) do all the hard work.

(a) (5 points) We want the transmitter to apply a transformation T to the n-length signal we want to communicate, $\vec{s}$, such that after the transmitted signal $\vec{x} = T\vec{s}$ goes through the channel with circulant echo matrix H, the receiver sees $\vec{y} = \vec{s}$ at its antenna. **Assuming that T exists, what is this transformation T?**



Solution: The received vector is $\vec{y} = H\vec{x}$, and since $\vec{x} = T\vec{s}$, we have

$$\vec{y} = HT\vec{s}$$

Recall, we want to have $\vec{y} = \vec{s}$ for all $\vec{s}$. That is,

$$(HT)\vec{s} = \vec{s}$$

for all $\vec{s}$. A necessary and sufficient condition for this is $HT = I$, or $\boxed{T = H^{-1}}$.

(b) (8 points) **Assuming that T exists, what are the eigenvectors of T, and what are the eigenvalues for T in terms of the eigenvalues of H?**

(Hint: think about diagonalizing H and looking in that basis. Also recall $(AB)^{-1} = (B^{-1}A^{-1})$ if these are square matrices.)

Solution: H is a circulant matrix, so we know that its eigenvectors are the DFT basis of length n. These are the normalized complex exponentials where the k-th one has t-th component $u_k[t] = \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n}kt}$. Then we can diagonalize H as

$$H = P\Lambda P^{-1}$$

where the columns of P are the DFT basis. From the previous section we have that $T = H^{-1}$. Then applying the hint we have

$$\begin{aligned} T &= H^{-1} \\ &= \left(P\Lambda P^{-1}\right)^{-1} = \left((P\Lambda)P^{-1}\right)^{-1} \\ &= P(P\Lambda)^{-1} \\ &= P\Lambda^{-1}P^{-1} \end{aligned}$$

Then T is diagonalized under the same basis as H. The eigenvectors of T are the same as the eigenvectors of H, which are the DFT basis vectors. The eigenvalues of T are the reciprocals of the eigenvalues of H.

(c) (5 points) **Give a condition on λ_ℓ (an eigenvalue of H) such that if even one of the λ_ℓ satisfies this condition, then T cannot exist and it is impossible to get $\vec{y} = \vec{s}$ in general.**

(This means that at least some of the information in $\vec{s}$ is doomed to be lost forever.)

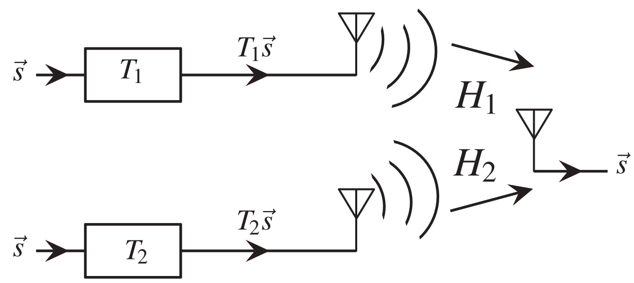
Solution: T does not exist if and only if H is non-invertable. By the previous part (writing $H = P\Lambda P^{-1}$), this occurs if and only if Λ is non-invertable. And since Λ is diagonal, it is non-invertable iff some diagonal entry $\boxed{\lambda_\ell = 0}$. In general, a diagonalizable matrix is non-invertable iff some eigenvalue is 0.

PRINT your name and student ID: ________________________________

(d) (15 points) We decide to add a second antenna to our transmitter. These two antennas will broadcast $T_1\vec{s}$ and $T_2\vec{s}$. Suppose $n = 3$ in this part.

The antennas are far apart and so their echo patterns are different. Antenna 1 has circulant echo matrix H_1 with first column $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ and antenna 2 has circulant echo matrix H_2 with first column $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The receiver receives the 3-long $\vec{y} = H_1T_1\vec{s} + H_2T_2\vec{s}$ which is the sum of the two transmitted signals with their respective echoes.



We can diagonalize H_1 as follows

$$H_1 = \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{bmatrix} \right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{bmatrix} \right)^{-1}.$$

- **First, write out H_2 and diagonalize it.**
- **And then, give matrices T_1 and T_2 so that $\vec{y} = \vec{s}$.**

It suffices to give T_1 in the eigenbasis for H_1 and T_2 in the eigenbasis for H_2. There are multiple right answers here but we encourage you to just pick whichever you consider the simplest.

Solution: The first column of H_2 is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, which is the same for every circular shift, so

$$H_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

.

We know that the eigenbasis of every circulant matrix is the DFT basis, so we can write H_2 as

$$H_2 = \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{bmatrix} \right) \Lambda_2 \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{bmatrix} \right)^{-1}$$

To find the eigenvalues associated with each eigenvector, we compute $H_2\vec{p}_i$ for each basis vector $\vec{p}_i$.

$$
\begin{aligned}
H_2\vec{p}_1 &= H_2 \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix} = 3\vec{p}_1 \\
H_2\vec{p}_2 &= H_2 \begin{bmatrix}1\\e^{\frac{2\pi}{3}}\\e^{\frac{4\pi}{3}}\end{bmatrix} = \left(1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} \\
&= \left(1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \\
H_2\vec{p}_3 &= H_2 \begin{bmatrix}1\\e^{\frac{4\pi}{3}}\\e^{\frac{8\pi}{3}}\end{bmatrix} = \left(1 + e^{i\frac{4\pi}{3}} + e^{i\frac{8\pi}{3}}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} \\
&= \left(1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}
\end{aligned}
$$

Then the eigenvector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ has eigenvalue 3, and the other eigenvectors have eigenvalue 0. Then you can write H_2 as

$$
H_2 = \left(\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}}\\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}}\end{bmatrix}\right)\begin{bmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\left(\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}}\\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}}\end{bmatrix}\right)^{-1}
$$

Let P be the DFT basis, and let the diagonalizations of H_1, H_2 be $H_1 = P\Lambda_1 P^{-1}$ and $H_2 = P\Lambda_2 P^{-1}$. Now, suppose we consider matrices T_1, T_2 in the DFT basis, as $T_1 = PM_1P^{-1}$ and $T_2 = PM_2P^{-1}$, for some matrices M_1, M_2.

Now that we have diagonalized everything, let us write what we want to achieve:

$$
\begin{aligned}
\vec{s} &= \vec{y} \\
\vec{s} &= H_1T_1\vec{s} + H_2T_2\vec{s} \\
&= (H_1T_1 + H_2T_2)\vec{s} \\
&= (P\Lambda_1M_1P^{-1} + P\Lambda_2M_2P^{-1})\vec{s} \\
&= P(\Lambda_1M_1 + \Lambda_2M_2)P^{-1}\vec{s}
\end{aligned}
$$

Left-multiplying by P^{-1}:

$$
(P^{-1}\vec{s}) = (\Lambda_1M_1 + \Lambda_2M_2)(P^{-1}\vec{s})
$$

for all $\vec{s}$. Thus, a neccesary and sufficient condition for this is

$$
\Lambda_1M_1 + \Lambda_2M_2 = I.
$$

Recall that in our case:

$$\Lambda_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \Lambda_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, if we want $\Lambda_1 M_1 + \Lambda_2 M_2 = I$, we can choose for example:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Recall that these M_1, M_2 define T_1, T_2 in their eigenbasis, and so $T_1 = PM_1P^{-1}$, $T_2 = PM_2P^{-1}$).
We remark that there are in fact many choices of M_1, M_2 here. For starters, we can notice that anything of the form:

$$M_1 = \begin{bmatrix} \star & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$$

works, where $(\star)$ can be anything.
In particular, $T_1 = T_2 = \frac{1}{3}I$ works.

12. One Magical Procedure (27+5 points)

Suppose we have a vector $\vec{x} \in \mathbb{R}^5$ and an $n \times 5$ measurement matrix M defined by column vectors $\vec{c}_1, \cdots, \vec{c}_5$ such that:

$$M\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^n$ as a noisy measurement of the vector $\vec{x}$, with measurement matrix M and some additional noise in it as well.

You also know that the **true $\vec{x}$ is sparse — it only has two non-zero entries and all the rest of the entries are zero in reality.** Our goal is to recover this original $\vec{x}$ as best we can.

However, your intern has managed to lose not only the measurements $\vec{b}$ but the entire measurement matrix M as well!

Fortunately, you have found a backup in which you have all the pairwise inner-products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of M and each other, as well as all the inner products $\left\langle \vec{c}_i, \vec{b} \right\rangle$ between the columns of M and the vector $\vec{b}$. Finally, you also find the inner-product $\left\langle \vec{b}, \vec{b} \right\rangle$ of $\vec{b}$ with itself.

All the information you have is captured in the following table of inner-products. (These are not the vectors themselves.)

$\langle .,. \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, the inner product $\left\langle \vec{c}_3, \vec{b} \right\rangle = 2$, and that the inner product $\left\langle \vec{b}, \vec{b} \right\rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of $\vec{x}$ are non-zero, and what their values are.

(a) (4 points) **Use the information in the table above to answer which of the $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with $\vec{b}$?**

Solution:

Reading off the table, $\boxed{\vec{c}_4}$ has the largest inner product with $\vec{b}$.

(b) (5 points) Let the vector with the largest magnitude inner product with $\vec{b}$ be $\vec{c}_a$. Let $\vec{b}_p$ be the projection of $\vec{b}$ onto $\vec{c}_a$. **Write $\vec{b}_p$ symbolically as an expression only involving $\vec{c}_a$, $\vec{b}$ and their inner-products with themselves and each other.**

Solution:

The magnitude of the projection is $\frac{\langle \vec{c}_a, \vec{b} \rangle}{||\vec{c}_a||}$, and the direction of the projection is $\frac{\vec{c}_a}{||\vec{c}_a||}$. Thus:

$$\vec{b}_p = \boxed{\frac{\left\langle \vec{c}_a, \vec{b} \right\rangle}{\langle \vec{c}_a, \vec{c}_a \rangle} \vec{c}_a}$$

(c) (10 points) **Use the information in the table above to find which of the column vectors $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.**

(Hint: the linearity of inner products might prove useful.)

Solution: The inner product of $\vec{b} - \vec{b}_p$ with a vector $\vec{c}_1$ is:

$$\left\langle \vec{b} - \vec{b}_p, \vec{c}_1 \right\rangle = \left\langle \vec{b}, \vec{c}_1 \right\rangle - \frac{\left\langle \vec{c}_a, \vec{b} \right\rangle}{\langle \vec{c}_a, \vec{c}_a \rangle} \langle \vec{c}_a, \vec{c}_1 \rangle$$

Finding the numerical values of the inner products:

$\left\langle \vec{b} - \vec{b}_p, \vec{c}_1 \right\rangle$	$\left\langle \vec{b} - \vec{b}_p, \vec{c}_1 \right\rangle$	$\left\langle \vec{b} - \vec{b}_p, \vec{c}_1 \right\rangle$	$\left\langle \vec{b} - \vec{b}_p, \vec{c}_1 \right\rangle$	$\left\langle \vec{b} - \vec{b}_p, \vec{c}_1 \right\rangle$
4	-2	2	0	2

Thus the vector with the highest inner product with the residue is: $\boxed{\vec{c}_1}$

(d) (8 points) Suppose the vectors we found in parts (a) and (c) are $\vec{c}_a$ and $\vec{c}_c$. These correspond to the components of $\vec{x}$ that are non-zero, that is, $\vec{b} \approx x_a\vec{c}_a + x_c\vec{c}_c$. However, there might be noise in the measurements $\vec{b}$, so we want to find the linear-least-squares estimates $\widehat{x}_a$ and $\widehat{x}_c$. **Write a matrix expression for $\begin{bmatrix}\widehat{x}_a\\ \widehat{x}_c\end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a$, $\vec{c}_c$, $\vec{b}$.**

Solution:

We use least-squares to solve for $\begin{bmatrix}\widehat{x}_a\\ \widehat{x}_c\end{bmatrix}$. Let $A = \begin{bmatrix}\vec{c}_a & \vec{c}_c\end{bmatrix}$. Using the least-squares formula,

$$\begin{bmatrix}\widehat{x}_a\\ \widehat{x}_c\end{bmatrix} = (AA^T)^{-1}A^T\vec{b}$$

$$= \begin{bmatrix}\langle\vec{c}_a,\vec{c}_a\rangle & \langle\vec{c}_a,\vec{c}_c\rangle\\ \langle\vec{c}_c,\vec{c}_a\rangle & \langle\vec{c}_c,\vec{c}_c\rangle\end{bmatrix}^{-1}\begin{bmatrix}\left\langle\vec{c}_a,\vec{b}\right\rangle\\ \left\langle\vec{c}_c,\vec{b}\right\rangle\end{bmatrix}$$

(e) (BONUS: 5 points) **Compute the numerical values of $\widehat{x}_a$ and $\widehat{x}_c$ using the information in the table.**

Solution: Substituting the previous expression with values from the table, we get: $x_1 = 2\frac{2}{3}, x_4 = 4\frac{1}{3}$

$$\begin{bmatrix}\widehat{x}_4\\ \widehat{x}_1\end{bmatrix} = \begin{bmatrix}\langle\vec{c}_4,\vec{c}_4\rangle & \langle\vec{c}_4,\vec{c}_1\rangle\\ \langle\vec{c}_1,\vec{c}_4\rangle & \langle\vec{c}_1,\vec{c}_1\rangle\end{bmatrix}^{-1}\begin{bmatrix}\left\langle\vec{c}_4,\vec{b}\right\rangle\\ \left\langle\vec{c}_1,\vec{b}\right\rangle\end{bmatrix}$$

$$= \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}^{-1}\begin{bmatrix}6\\ 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}6\\ 1\end{bmatrix} = \begin{bmatrix}13/3\\ 8/3\end{bmatrix}$$

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Copy of table for convenience:

$\langle .,. \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

PRINT your name and student ID: ____

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed. Or tell us a joke or a story.]