EECS 16A Designing Information Devices and Systems I
Fall 2021 Final

1. A Quirky Quantum Question

(a) In quantum mechanics, states of particles are represented by vectors in a vector space. In this problem, we'll say that all states exist in $\mathbb{R}^2$.

A particular matrix, $\hat{\mathbf{H}}$ (called the Hamiltonian operator), has the unique property that its eigenvalues represent a particle's allowed energy values. Quantum mechanics tells us that if the values of $\hat{\mathbf{H}}$ are real, it must be symmetric – that is, it can be written as

$$\hat{\mathbf{H}} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Assume that we know $a > 0$ and $b > 0$. What further condition on a and b forces the allowed energy values (the eigenvalues) to always be nonnegative?

Solution: We must find the eigenvalues of $\hat{\mathbf{H}}$ and determine a condition that forces them to be nonnegative.

$$\hat{\mathbf{H}}\vec{x} = \lambda\vec{x}$$
$$\hat{\mathbf{H}}\vec{x} - \lambda\mathbf{I}\vec{x} = \vec{0}$$
$$(\hat{\mathbf{H}} - \lambda\mathbf{I})\vec{x} = \vec{0}$$
$$\begin{vmatrix} (a-\lambda) & b \\ b & (a-\lambda) \end{vmatrix} = 0$$
$$(a-\lambda)^2 - b^2 = 0$$
$$(a-\lambda) = \pm b$$
$$\lambda = a \pm b$$

Since a and b are both positive, $a+b$ will never produce a negative eigenvalue. However, in order for $a-b$ to be nonnegative, $a \geq b$.

(b) Miki experimentally determines that particles associated with the $\hat{\mathbf{H}}$ matrix from Question 1 have allowed energy values $\lambda_1 = \frac{5}{2}$ and $\lambda_2 = \frac{9}{2}$. Find a and b.

Solution: We know from the first question that $\lambda = a \pm b$. Since $\lambda_2 > \lambda_1$ and a and b are both positive, $\lambda_2 = a+b$ and $\lambda_1 = a-b$. Thus, we have two equations and two unknowns:

$$a - \lambda_1 = b \text{ and } \lambda_2 - a = b$$
$$a - \lambda_1 = \lambda_2 - a$$
$$a = \frac{1}{2}(\lambda_2 + \lambda_1)$$
$$a = \frac{1}{2}(\frac{9}{2} + \frac{5}{2})$$
$$a = \frac{7}{2} = 3.5$$

Likewise, we can use the same process to solve for b:

$$\begin{aligned}\lambda_1 + b = a \text{ and } \lambda_2 - b = a\\ \lambda_1 + b = \lambda_2 - b\\ b = \frac{1}{2}(\lambda_2 - \lambda_1)\\ b = \frac{1}{2}(\frac{9}{2} - \frac{5}{2})\\ b = 1\end{aligned}$$

(c) Now, given a new matrix $\hat{\mathbf{H}} = \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}$, let the eigenvalues be $\lambda_1 < \lambda_2$ and the normalized eigenvectors be $\vec{v_{\lambda_1}}$ and $\vec{v_{\lambda_2}}$ (corresponding to eigenvalues λ_1 and λ_2, respectively, and scaled to a magnitude of 1), span $\mathbb{R}^2$. If a particle is in some state $\vec{v_s} \in \mathbb{R}^2$, then it can be expressed as $\vec{v_s} = \alpha\vec{v_{\lambda_1}} + \beta\vec{v_{\lambda_2}}$, where α and β are real constants.

If $v_s = \begin{bmatrix}0\\1\end{bmatrix}$, what are possible magnitudes of α? (In quantum mechanics, α^2 represents the probability that measuring the particle's energy will yield λ_1.)

Solution: The first step is to find the eigenvectors of $\hat{\mathbf{H}}$. Based on previous questions, we know that $\lambda_1 = a - b = 3 - 2 = 1$, and $\lambda_2 = a + b = 3 + 2 = 5$.

Let's define $\vec{v_{\lambda_1}} = \begin{bmatrix}c_1\\d_1\end{bmatrix}$ and $\vec{v_{\lambda_1}} = \begin{bmatrix}c_2\\d_2\end{bmatrix}$. We can now produce conditions on c_1, d_1, c_2, and d_2 to find the eigenvectors:

$$\begin{aligned}\hat{\mathbf{H}}\vec{v_{\lambda_1}} &= \lambda_1\vec{v_{\lambda_1}}\\ \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}c_1\\d_1\end{bmatrix} &= \begin{bmatrix}c_1\\d_1\end{bmatrix}\\ \begin{bmatrix}3c_1 + 2d_1\\2c_1 + 3d_1\end{bmatrix} &= \begin{bmatrix}c_1\\d_1\end{bmatrix}\end{aligned}$$

We can take the first row in isolation to produce

$$\begin{aligned}3c_1 + 2d_1 &= c_1\\ c_1 &= -d_1\end{aligned}$$

As such, we know that the first eigenvector will have the form $\vec{v_{\lambda_1}} = k_1\begin{bmatrix}1\\-1\end{bmatrix}$ for some normalization constant k_1. Since we're told that $|\vec{v_{\lambda_1}}| = 1$, k_1 must equal $\frac{1}{\sqrt{2}}$. Thus, $\vec{v_{\lambda_1}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$.

Now, we can produce another condition using the second eigenvalue:

$$\begin{aligned}\hat{\mathbf{H}}\vec{v_{\lambda_2}} &= \lambda_2\vec{v_{\lambda_2}}\\ \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}c_2\\d_2\end{bmatrix} &= 5\begin{bmatrix}c_2\\d_2\end{bmatrix}\\ \begin{bmatrix}3c_2 + 2d_2\\2c_2 + 3d_2\end{bmatrix} &= 5\begin{bmatrix}c_2\\d_2\end{bmatrix}\end{aligned}$$

We can once again take the first row in isolation to produce

$$\begin{aligned} 3c_2 + 2d_2 &= 5c_2 \\ c_2 &= d_2 \end{aligned}$$

As such, we know that the second eigenvector will have the form $\vec{v_{\lambda_2}} = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some normalization constant k_2. Since we're told that $|\vec{v_{\lambda_2}}| = 1$, k_2 must equal $\frac{1}{\sqrt{2}}$. Thus, $\vec{v_{\lambda_2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The last step is to find α. Based on the problem statement,

$$\begin{aligned} \alpha \vec{v_{\lambda_1}} + \beta \vec{v_{\lambda_2}} &= \vec{v_s} \\ \alpha \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Since we know that $\vec{v_{\lambda_1}}$ and $\vec{v_{\lambda_2}}$ are linearly independent, this equation has a unique solution. α and β may be found through row reduction or inversion, but by inspection we can see that $\alpha = -\frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{\sqrt{2}}$. Thus, $|\alpha| = \frac{1}{\sqrt{2}}$.

2. Inner Products

(a) For the following inner product defined on $\mathbb{R}^2$, which inner product properties hold?

$$< \vec{x}, \vec{y} >= \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3\vec{x} + 3\vec{y})$$

i. Symmetry (True/False)
ii. Linearity (True/False)
iii. Positive-Definiteness (True/False)

Solution:

i. Symmetry TRUE. Note that we have:

$$< \vec{x}, \vec{y} >= \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3\vec{x} + 3\vec{y}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3\vec{y} + 3\vec{x}) =< \vec{y}, \vec{x} >$$

so $< \vec{x}, \vec{y} >=< \vec{y}, \vec{x} >$ and symmetry holds.

ii. Linearity FALSE. Let $c \in \mathbb{R}$:

$$< c\vec{x}, \vec{y} >= \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3c\vec{x} + 3\vec{y})$$

However,

$$c < \vec{x}, \vec{y} >= c \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3\vec{x} + 3\vec{y}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3c\vec{x} + 3c\vec{y})$$

Therefore, $< c\vec{x}, \vec{y} >\neq c < \vec{x}, \vec{y} >$ and linearity does not hold.

iii. Positive-Definiteness FALSE. Let $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. We have:

$$<\vec{x},\vec{x}> = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T (3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T \begin{bmatrix} -6 \\ -6 \end{bmatrix} = -18$$

Therefore,

$$<\vec{x},\vec{x}> < 0$$

and positive-definiteness does not hold.

(b) Given the following valid inner product over the vector space of 2x2 real matrices $\mathbb{R}^{2x2}$, defined as $<A,B> = <\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}> = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$ for any $A, B \in \mathbb{R}^{2x2}$. Given this inner product definition, what is $\| \begin{bmatrix} 2 & 5 \\ 6 & 2 \end{bmatrix} \|^2$ (or alternate exam values $\| \begin{bmatrix} 7 & 1 \\ 0 & 3 \end{bmatrix} \|^2$?

Solution: Remember that for any inner product space, $\|A\|^2 = <A,A>$. Knowing that, we have:

$$\| \begin{bmatrix} 2 & 5 \\ 6 & 2 \end{bmatrix} \|^2 = < \begin{bmatrix} 2 & 5 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 6 & 2 \end{bmatrix} > = 2^2 + 5^2 + 6^2 + 2^2 = 69$$

Thus, the answer is 69.
Using similar procedures with the alternate numbers, the answer is 59.

3. Least Squares with Shazam

(a) The application Shazam is able to detect what song is playing by means of an *acoustic footprint*. This is a small set of information that identifies the song. Shazam then checks that footprint in its database, to check for another song that has that footprint.

Here is the footprint we obtained via sampling: (we are representing the footprint as a vector)

$$\vec{x}_{sample} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad (1)$$

Say Shazam has narrowed it down to the following three songs with the corresponding footprints:

"Electric Love - Børns": $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

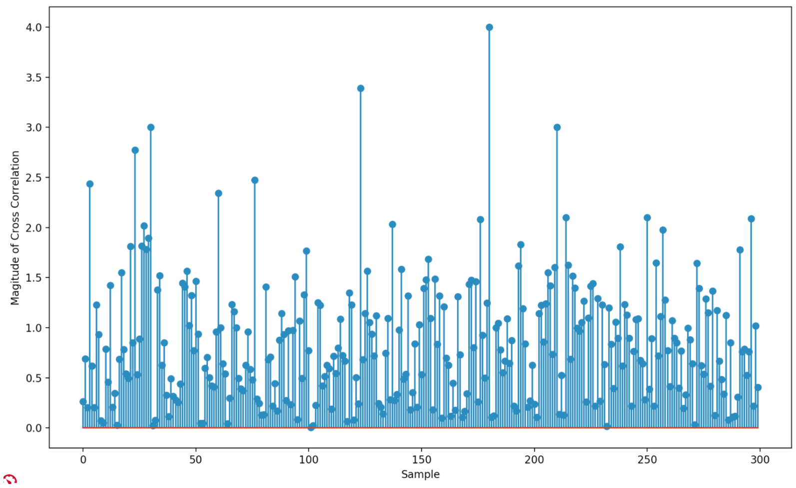
"She's Electric - Oasis": $\vec{x}_2 = \begin{bmatrix} 2 \\ -2 \\ -8 \\ 7 \end{bmatrix}$

"Electric Feel - MGMT": $\vec{x}_3 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 2 \end{bmatrix}$

Shazam is going to determine which song it is by projecting the footprint of our sample onto each of the song candidates, and ranking the songs based on the normalized inner product of $\vec{x}_{sample}$ onto the footprints.
Based on this information, which song is playing?

Solution: Correct Answer: Electric Feel. The formula for projection of $\vec{u}$ onto $\vec{v}\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\vec{v}$, but we only care about the magnitude of this vector, in other words, $\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}$. Magnitudes of projection are 0, $\frac{19}{11}$, $\frac{12}{5}$ respectively, the song which has the projection of largest magnitude is the one that is the most similar.

(b) Shazam has a feature where it displays lyrics in time with the song that is playing - to do this, it needs to figure out where in the song we are. To do this, it takes the cross-correlation of a snippet of the song with the full song. Assume that the song is sampled at 30 samples a second. Here is the cross correlation:



Make the best estimate as to when the sample was taken in seconds.

Solution: 6. We are looking for the largest positive spike, as this indicates where the signal is most similar to the sample. This is achieved at 180 samples, or 6 seconds.

(c) Shazam wants to partner with Spotify. For its Discover Weekly algorithms, Spotify would like to know what characteristics of a song make it attractive to a first-time listener. Shazam provides the number of Shazams for a set of new songs to Spotify, who combines it with their data on the songs. Here is the table of data that Spotify assembles:

Shazam Popularity	Tempo	Danceability	Acoustic-ness
1109	100	0.8	0.6
5501	90	0.5	0.2
2031	68	0.4	0.7
13045	120	0.9	0.2

Spotify would now like to use the data it has to predict what the number of shazams for a new song, whose characteristics (Tempo, Danceability, Acoustic-ness) are represented as $\vec{a}_n$.

Which is the correct formula for how Spotify would use Least Squares to calculate this? Let M be the matrix of Tempo, Danceabilty and Acoustic-ness, and $\vec{b}_n$ be the number of shazams they get:

$$M = \begin{bmatrix} 100 & 90 & 68 & 120 \\ 0.8 & 0.5 & 0.4 & 0.9 \\ 0.6 & 0.2 & 0.7 & 0.2 \end{bmatrix}, \vec{b}_n = \begin{bmatrix} 1109 \\ 5501 \\ 2031 \\ 13045 \end{bmatrix}$$

Options:

i. $(MM^T)^{-1}M\vec{b}_n$

ii. $\vec{a}_n{}^T(MM^T)^{-1}M\vec{b}_n$

iii. $\vec{a}_n{}^T(\vec{b}_n{}^T M^T)^T$

iv. $\vec{b}_n{}^T M\vec{a}_n$

v. $\vec{b}_n{}^T M\vec{a}_n$

[0.1in] **Solution:** $\vec{a}_n{}^T(MM^T)^{-1}M\vec{b}_n$. To find the weights we do $\vec{x}_{approx} = (MM^T)^{-1}M\vec{b}_n$. This boils down to finding the weights that minimize error, giving us the best predictor possible. We then want to run our new data through the model we just made, so we do: $\vec{a}_n{}^T\vec{x}_{approx}$to get our final answer. $(A^TA)^{-1}A^T\vec{b}_n$ is the way we do least squares, but in this case, the A matrix we want to plug into the expression is M^T. This is because M^T has the data for each characteristic in the columns (each characteristic is one column). Remember that in least squares, we want to find the way in which we should combine the columns in order to best approximate the $\vec{b}$ vector. If we use M^T, we end up finding what linear combination of characteristics best approximates the number of shazams, which is exactly what we want to do - predict Shazams popularity based on the characteristics. $\vec{x}_{approx} = (MM^T)^{-1}M\vec{b}_n$ is not the answer itself, it is just tells us how we should weight the characteristics - we need to run our new data through the model we just made.

(d) Say Spotify gets some new data to incorporate into its data set, the energy of the song. Here is the table with the added data:

Shazam Popularity	Tempo	Danceability	Acoustic-ness	Energy
1109	100	0.8	0.6	0.70
5501	90	0.5	0.2	0.35
2031	68	0.4	0.7	0.55
13045	120	0.9	0.2	0.55

Will it still be possible to run least squares with all of this data?

Solution: The Energy column is linearly dependant on the dancabilty and acoustic-ness - it is their average. This means that the columns of our least squares matrix would be dependant, and this means that least squares cannot work - this is because we have to calculate $\left(A^TA\right)^{-1}$. If the columns of A are linearly dependant, then A^TA will not be invertable, breaking our least squares formula. For a more intuitive reason, consider that least squares attempts to answer the question - how do I best linearly combine the columns of A so as to best approximate $\vec{b}$ If the columns of A are dependant, then there would be more than one way to linearly combine the column vectors of A to arrive at the best estimate of $\vec{b}$.

(e) What is the maximum number of features we could have per song, assuming we keep the number of songs the same?

Options:

i. 4

ii. 3

iii. 2

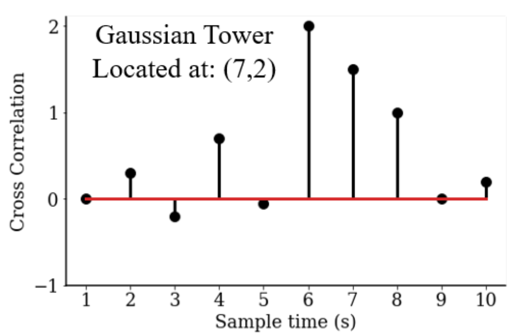
iv. 5

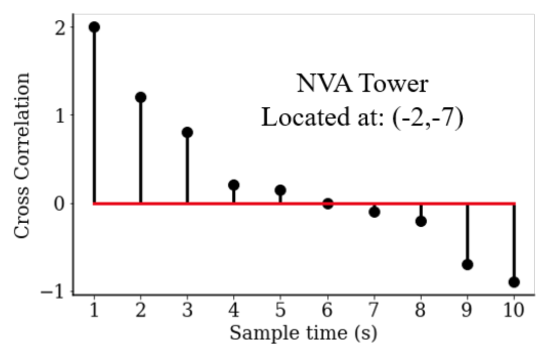
v. As many as we want

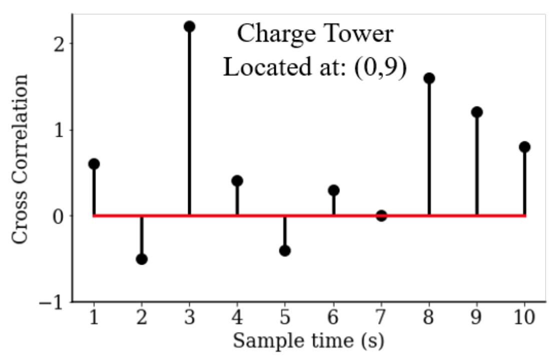
Solution: The maximum number of features is 4. This is because in order for least squares to work, we need to have linearly independent columns, which are of dimension of 4. One can have, at maximum, a set of n vectors in R^n that are all linearly independent, so the maximum number of features is 4.

4. Towers of Sixteen

The kingdom of Sixteen is easy to get lost in. Luckily, there are three beacons that send out audio signals to help lost travelers. You have a special device that records one sample every second and plots the cross correlation of the known signals emitted from each of these beacons with the signal it has received. After you begin recording the signals, you see the following three plots, associated with the three beacons named Gaussian, NVA, and Charge:







(a) How far away are you from each of the beacons? Provide a distance in meters. Hint: Approximate the speed of sound to 300 meters per second and consider what information the cross-correlation plots can provide.

i. Distance from the Gaussian Tower:[ ]meters

ii. Distance from the NVA Tower:[ ]meters

iii. Distance from the Charge Tower:[ ]meters

You write a system of linear equations based on the data you have collected to help you find your location in the 2D space of the kingdom. To solve this system, you decide to use Gaussian Elimination and set up a matrix-vector equation in the form: $\mathbf{A}\vec{x} = \vec{b}$.

Which matrix or vector corresponds the most closely to the distances you found above correspond with? Answer with A, $\vec{x}$, or $\vec{b}$

Solution: We know that distance (meters) is equal to speed (meters/second) multiplied by time (seconds). We are given the approximate speed of sound in the hint, so we need to find out how long it took the signal from each of the towers to reach us for the time so we can then calculate the distances we need. The cross-correlation of our received signal with each known signal tells us the similarity of the two at different times, so we can use the given cross-correlation graphs to find the delay of the signals from each tower. This will be where the two signals are the most similar, where the correlation is the highest. For the Gaussian Tower, we see this peak at 6 seconds, so we know it took the audio signal 6 seconds to reach us. Similarly, we received the signal from the NVA Tower 1 second after it was sent out and from the Charge Tower 3 seconds after it was sent out. Thus, our distance from each of the towers is:

Distance from the Gaussian Tower = 300 meters/second * 6 seconds = 1800 meters;

Distance from the NVA Tower = 300 meters/second * 1 second = 300 meters;

Distance from the Charge Tower = 300 meters/second * 3 seconds = 900 meters;

We are trying to find our location given the locations of some beacons and our distance from those beacons, so we can try utilizing trilateration. Each of our equations would be in the form:

$$(x_{unknown} - x_{beacon})^2 + (y_{unknown} - y_{beacon})^2 = Distance^2.$$

The distances we found in this question are the constants on right side of each equation, so they would be part of the $\vec{b}$, and thus, our answer would be "b". The coefficients of the expanded left side of each equation would yield each row entry in the “A” matrix, and your current location would be found in the “x” vector.

(b) **Version 1:** You continue to explore another area of the kingdom and now want to share your location with your friends so they can also join you in your travels! Regardless of what you got from the previous part, assume that you have already calculated the new distances between yourself and each of the beacons:

Distance from the Gaussian Tower: $\sqrt{18}$ meters

Distance from the NVA Tower: LaTeX: $\sqrt{72}$ meters

Distance from the Charge Tower: $\sqrt{116}$ meters

Then, your location is at ([x], [y]).

Version 2: You continue to explore another area of the kingdom and now want to share your location with your friends so they can also join you in your travels! Regardless of what you got from the previous part, assume that you have already calculated the new distances between yourself and each of the beacons:

Distance from the Gaussian Tower: $\sqrt{101}$ meters

Distance from the NVA Tower: LaTeX: $\sqrt{65}$ meters

Distance from the Charge Tower: $\sqrt{73}$ meters

Then, your location is at ([x, y]).

Solution:

Version 1: Given the distance from each of the beacons, we can write down the equation according to part a). We have:
Distance from Gaussian Tower:

$$(x-7)^2+(y-2)^2 = x^2+y^2-14x-4y+49+4 = 18.$$

Distance from NVA Tower:

$$(x+2)^2+(y+7)^2 = x^2+y^2+4x+14y+4+49 = 72.$$

Distance from Charge Tower:

$$(x)^2+(y-9)^2 = x^2+y^2-18y+81 = 116.$$

Subtract equation 2 from equation 1, we have:

$$18x+18y = 54.$$

Subtract equation 3 from equation 1, we have:

$$14x-14y = 70.$$

Solve the system of equations, we have $x = 4, y = -1$.

Version 2: Given the distance from each of the beacons, we can write down the equation according to part a). We have:
Distance from Gaussian Tower:

$$(x-7)^2+(y-2)^2 = x^2+y^2-14x-4y+49+4 = 101.$$

Distance from NVA Tower:

$$(x+2)^2+(y+7)^2 = x^2+y^2+4x+14y+4+49 = 65.$$

Distance from Charge Tower:

$$(x)^2+(y-9)^2 = x^2+y^2-18y+81 = 73.$$

Subtract equation 2 from equation 1, we have:

$$18x+18y = 36.$$

Subtract equation 3 from equation 1, we have:

$$14x-14y = -56.$$

Solve the system of equations, we have $x = -3, y = 1$.

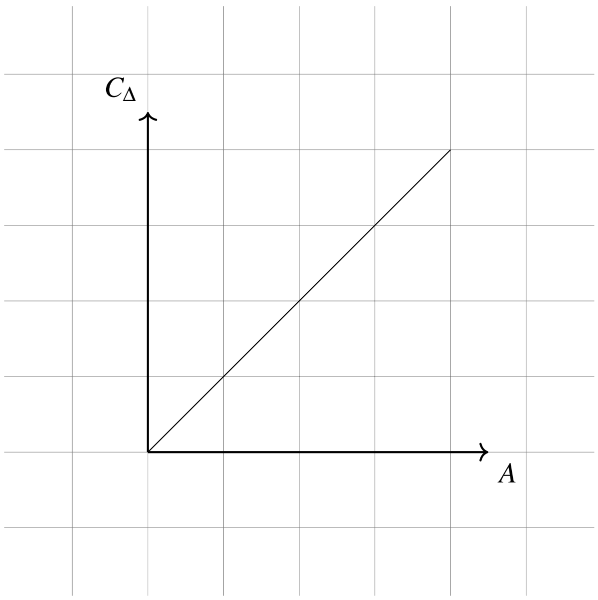
5. Capacitive Touch Pixel

In lab, we worked on a capacitive touch pixel that can detect whether a touch is present or not. Your TA Raghav wants to develop a capacitive touch pixel that can differentiate between no touch, weak touch, and strong touch. But he needs your help implementing this design.

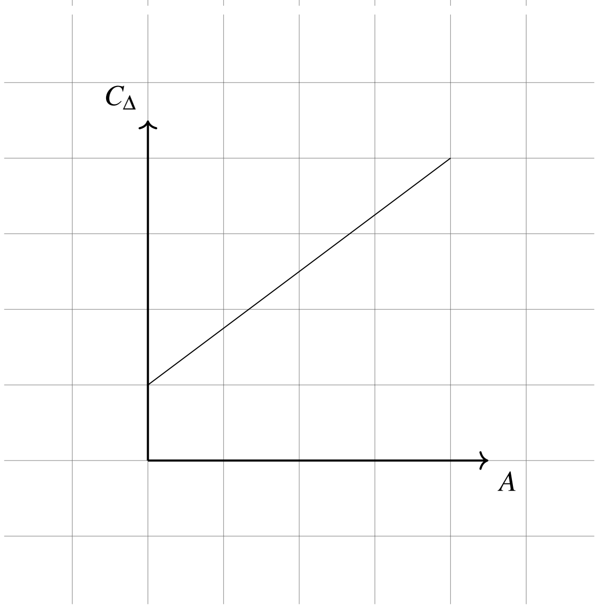
Just like lab, we model the surface of a finger as a parallel plate. Thus, our finger forms a set of parallel plate capacitors, with total capacitance C_Δ . The difference between a weak touch and a strong touch is the area of the finger surface (stronger touch $\rightarrow$ greater area).

(a) (1 point) Which of the following curves depicts the correct relationship between C_Δ and the area of the finger surface A?

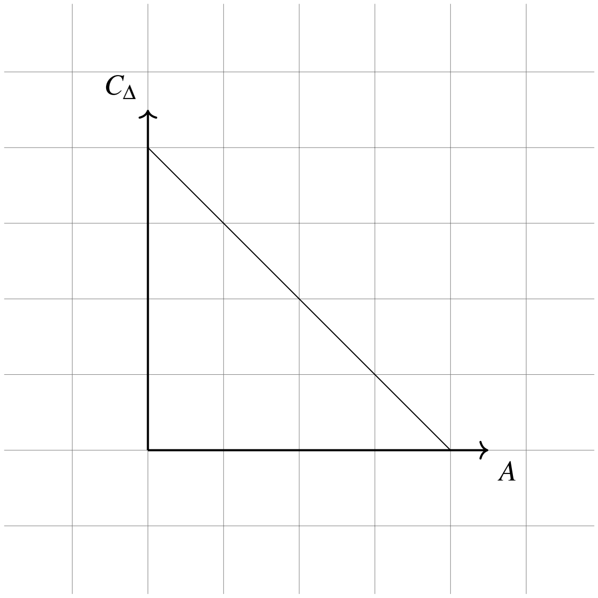
i.



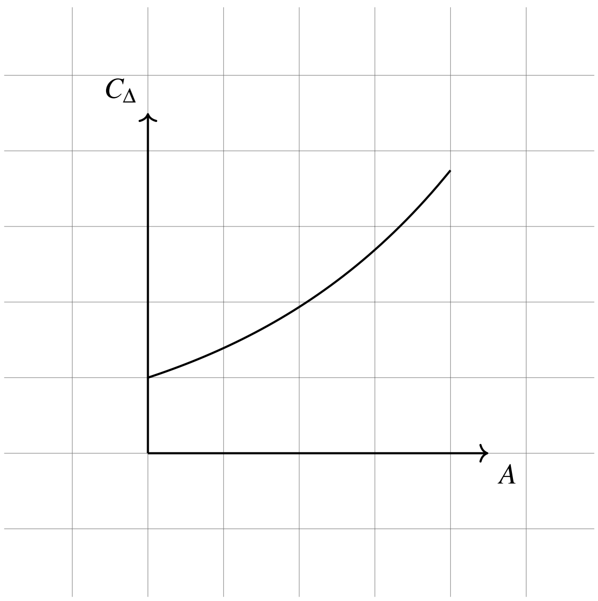
ii.

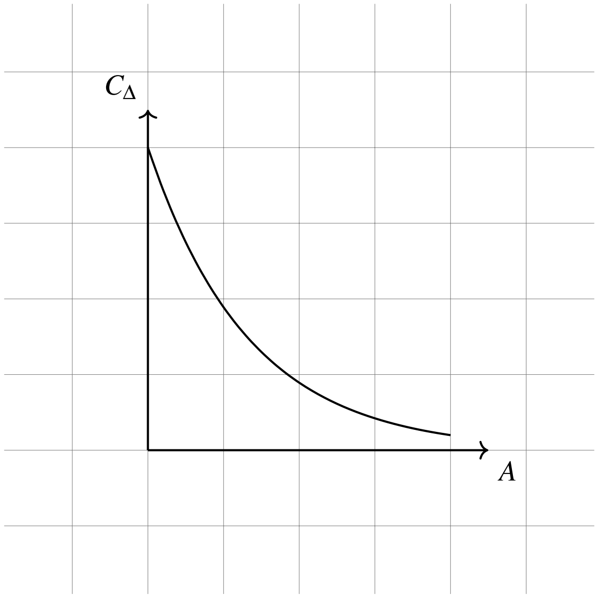


iii.

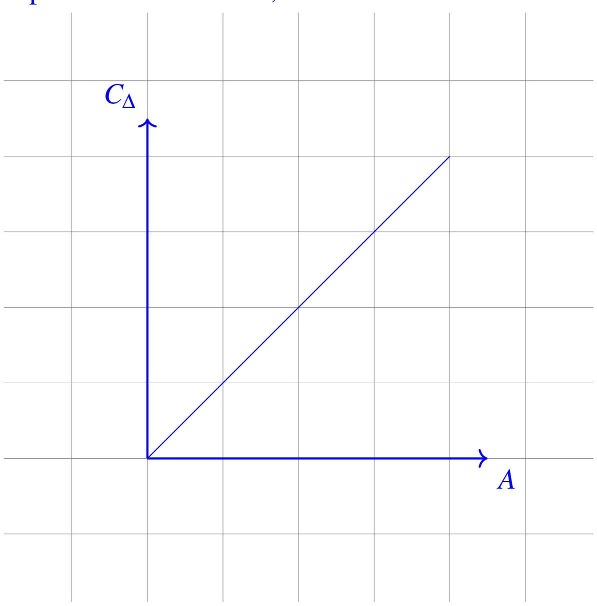


iv.



v. 

Solution: We know that for a parallel plate capacitor, $C \propto A$. When the area $A = 0$, there's no capacitance. Therefore, the correct answer is:



(b) (1 point) Let's say you're given $C_{\Delta,\text{weak touch}} = 5$ nF (3.5 nF for alternate version) and $A_{\text{strong touch}} =$

$1.4 \cdot A_{\text{weak touch}}$ ($3 \cdot A_{\text{weak touch}}$ for alternate version). What is $C_{\Delta,\text{strong touch}}$ in nF?

Solution: $\frac{C_{\Delta,\text{weak touch}}}{A_{\text{weak touch}}} = \frac{C_{\Delta,\text{strong touch}}}{A_{\text{strong touch}}}$

$\therefore C_{\Delta,\text{strong touch}} = C_{\Delta,\text{strong touch}} \cdot \frac{A_{\text{strong touch}}}{A_{\text{weak touch}}}$

For V1, $C_{\Delta,\text{weak touch}} = 5nF$, $A_{\text{strong touch}} = 1.4 \cdot A_{\text{weak touch}}$, and $C_{\Delta,\text{strong touch}} = 7nF$.

For V2, $C_{\Delta,\text{weak touch}} = 3.5nF$, $A_{\text{strong touch}} = 3 \cdot A_{\text{weak touch}}$, and $C_{\Delta,\text{strong touch}} = 10.5nF$.

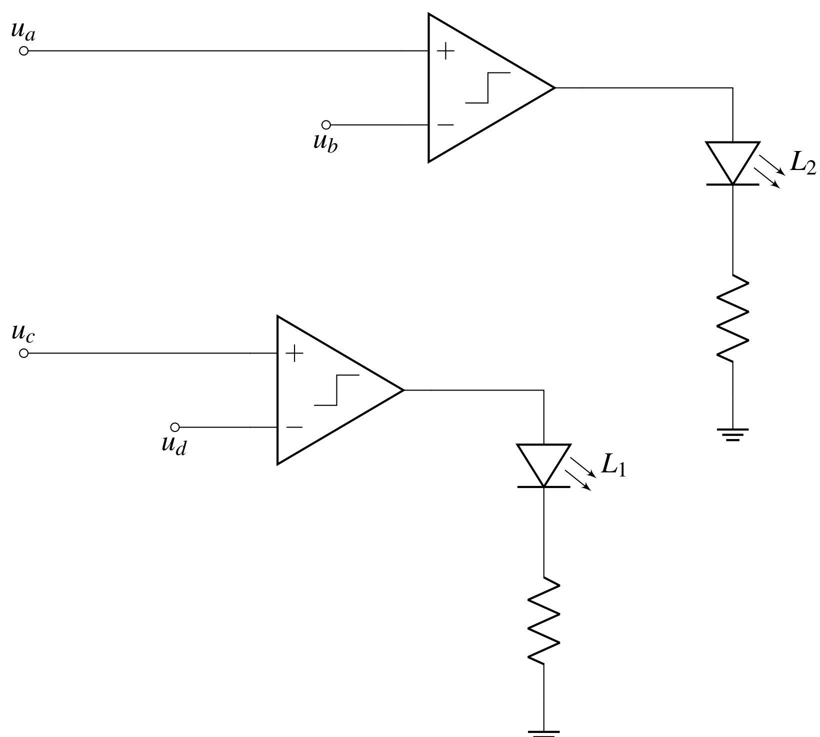
(c) (4 points) Let C_{pixel} be the capacitance at the touch pixel. Now we know that $C_{pixel} = C_0 + C_\Delta$ where C_0 is the capacitance of the pixel itself and C_Δ is the capacitance that comes from the finger. These capacitances will influence the node voltage V_+ after charge-sharing.

Raghav solves the charge-sharing problem for you and gives you the following information for the subsequent problem:

$V_{+,\text{no touch}} < V_{ref,1} < V_{+,\text{weak touch}} < V_{ref,2} < V_{+,\text{strong touch}}$

Choose the right voltages for the following nodes such that the given circuit takes V_+ as input and lights up LED L_1 when there is a weak touch and both LEDs L_1 and L_2 when there is a strong touch.

Assume the positive supply rail of the comparator is set at a large V_{DD} and the negative supply rail is set to ground ($0V$).



Find:

i. u_a

ii. u_b

iii. u_c

iv. u_d

Potential options:

i. V_+
ii. V_{DD}
iii. $V_{ref,1}$
iv. $0V$
v. $V_{ref,2}$
vi. $\frac{V_{DD}}{2}$

Solution: Since we want to compare whether the input V_+ is greater than a certain value to turn on one or both LEDs, the positive inputs to the comparators should be the input V_+.

$u_a = V_+$ and $u_c = V_+$

LED L_2 must only light up when we see a strong touch and not when we see a weak touch. So, $V_{+,\text{weak touch}} < u_b$ and $u_b < V_{+,\text{strong touch}}$.

$u_b = V_{ref,2}$

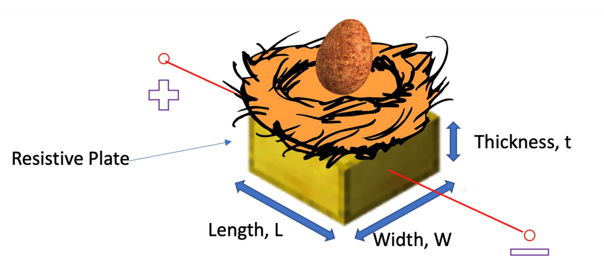
Similarly, LED L_1 must light up when we see a weak or a strong touch and not when we see no touch. So, $V_{+,\text{no touch}} < u_d$ and $u_d < V_{+,\text{weak touch}}$ (and $u_d < V_{+,\text{strong touch}}$).

$u_d = V_{ref,1}$

6. Falcon Incubation [8 points]

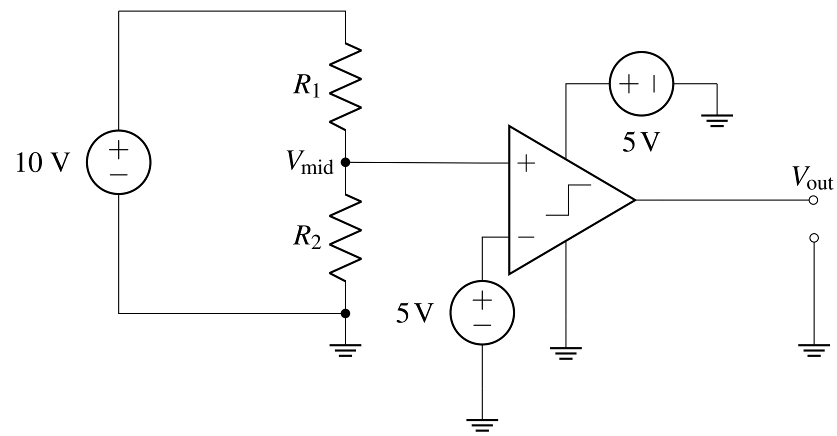
The Campanile, old as it is, needs to be cleaned sometimes. Unfortunately, this means our resident falcons will be displaced briefly. Being a conservation minded Berkeley Engineer, you decide to help them out by designing an artificial incubator. It has to hold the temperature at 37° C, and must turn on only once 3 eggs have been laid.

You decide to start by implementing a circuit that detects the number of eggs laid. For this, you use a resistive plate placed under the entire nest as shown in the following diagram:



We know that the plate resistance increases as more eggs are laid, essentially working as a variable resistor.

(a) **(1 point)** Your TA, Aniruddh, provides you with this circuit diagram which is designed to output 5V when 3 or more eggs have been lain, and 0 V otherwise.



If you have a single fixed resistor, R_{fixed}, and the variable resistance plate under the nest has a range such that $0 < R_{var} < 2R_{fixed}$, where should you put the resistive plate so that the circuit behaves as expected (with the fixed resistor in the other resistor location)?

i. The resistive plate must be placed where R_1 is.

ii. The resistive plate must be placed where R_2 is.

iii. The resistive plate can be placed in either location.

Solution: Remember that the resistance of our plate increases with the number of eggs lain. We therefore want u_+ to start at a value less than u_- when no eggs have been lain and increase to a value greater than or equal to u_- once 3 or more eggs have been lain. This should suggest that u_+, and therefore V_{mid} must be monotonically increasing with the number of eggs laid, and therefore the resistance of the plate. Notice that the V_{mid} is the middle node of a voltage divider, and therefore its voltage is:

$$V_{mid} = \frac{R_2}{R_1 + R_2} V_s$$

From this equation, we can see that V_{mid} is increases when $\mathbf{R}_2$ increases, and so this is where we should insert our plate.

(b) **(1 point)** Regardless of your previous answers, assume we use a fixed resistor for R_2 = 150Ω (or 100Ω for the alternate question). What is the maximum value of the other resistor such that the circuit outputs +5V (or operates on the threshold of the output voltage switch)? **Solution:** Recall that the circuit will switch when $u_+ \geq u_-$. Since u_- is held at 5V by a fixed voltage source, we therefore are looking for a value for R_1 which will set $V_{mid} = u_+ = 5V$. Recall the formula for a voltage divider that we used in part (a):

$$V_{mid} = \frac{R_2}{R_1 + R_2} V_s$$

We need to solve:

$$5 = \frac{150}{R_1 + 150} \times 10$$

This equation indicates that we need to set R_1 to also be **150** Ω.
Using a similar procedure, $R_1 = 100\Omega$ for the alternate question.

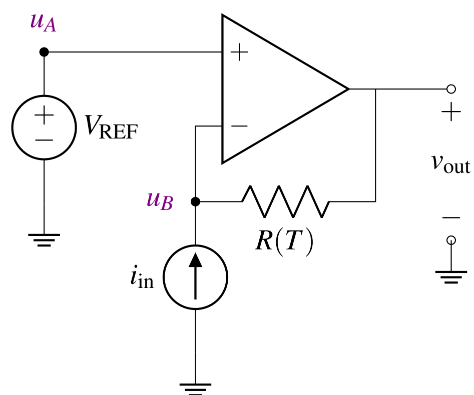
(c) **(4 points)** You now want to work on the temperature control of the incubation unit. You are provided with a *thermistor*, a temperature dependent resistor, which follows the following resistance (kΩ) vs. temperature (°C) relationship:

$$R(T) = 2.5 + \frac{1}{2}(T - 37) \qquad [k\Omega]$$

For the alternate question:

$$R(T) = 2 + \frac{1}{2}(T - 37) \qquad [k\Omega]$$

The thermistor can thus be used as a measurement of the current temperature. At 37°C, you want to output 0V to the temp controller–in other words, the below circuit should output 0V at this temperature.



If i_{in} = 4 mA (2 mA for the alternate question), what should V_{REF} be set to, in Volts (V)?

Solution: Notice that this is a *transresistance amplifier*. It is governed by the following formula:

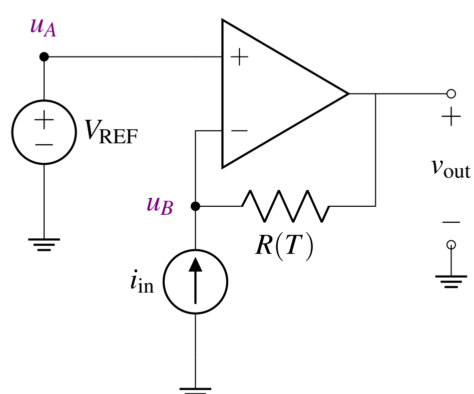
$$v_{out} = i_{in}(-R) + V_{REF}$$

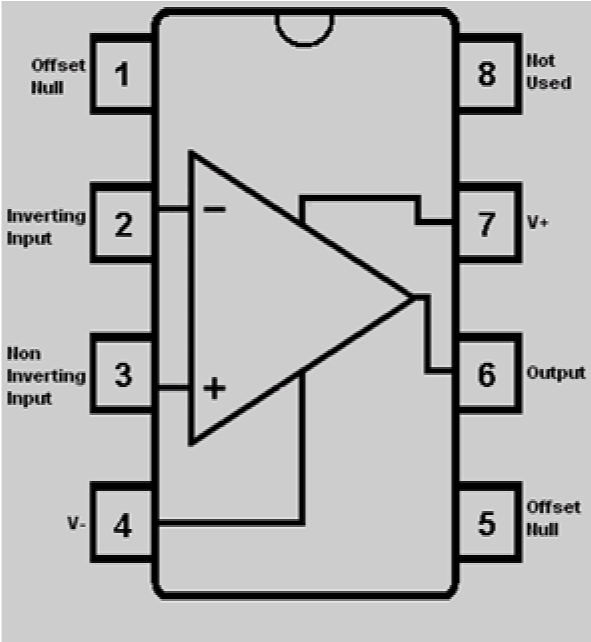
At 37 °C, notice that the thermistor has a resistance of 2.5 kΩ. Plugging these values in, and solving for 0V at 37 °C:

$$0V = 4 \times 10^{-3}(-2.5 \times 10^{3}) + V_{REF}$$
$$0V = -10V + V_{REF}$$

Therefore, V_{REF} should be set to **10 V**.
Using a similar procedure, V_{REF} should be set to **4 V** for the alternate question.

(d) **(2 points)** You are given the following op-amp pin diagram. Match the nodes from the previous circuit diagram (shown below for your convenience) to their corresponding pin numbers on the op-amp.





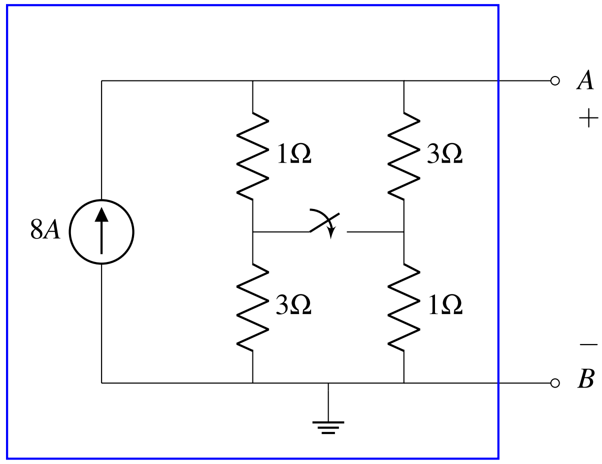
- u_B
- u_A
- 5V Supply
- -5V Supply
- V_{out}

Solution: Reading the labels on the Op-Amp pins gives us the following pairings:

- u_B : Pin 2
- u_A : Pin 3
- 5V Supply : Pin 7
- -5V Supply : Pin 4
- V_{out} : Pin 6

7. A Thevenin Paradox

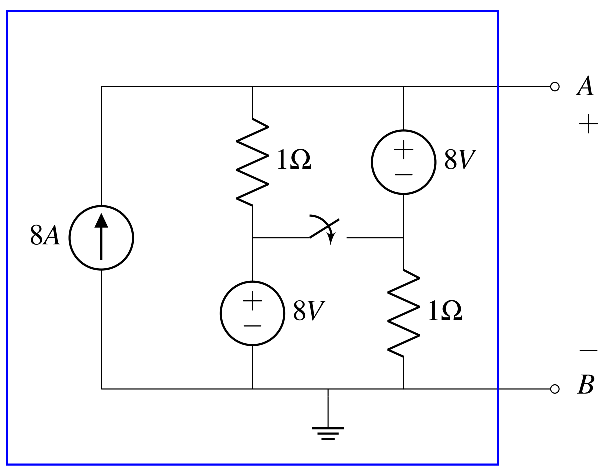
(a) Calculate the Thevenin equivalent voltage across terminals A and B first when the switch is open and again when the switch is closed.



Solution: When the switch is open, the circuit has two parallel branches each with a 3Ω and 1Ω resistor in series. So, each branch has an equivalent resistance of $3\Omega + 1\Omega = 4\Omega$. The total equivalent resistance is $R_{eq} = 4\Omega || 4\Omega = 2\Omega$. Hence, using Ohm's Law, the voltage across terminals A and B is $8A \times R_{eq} = 8A \times 2\Omega = 16V$.

When the switch is closed, there are two $(3\Omega || 1\Omega)$ resistors in series. Hence, the total equivalent resistance is $R_{eq} = 3\Omega || 1\Omega + 3\Omega || 1\Omega = 0.75\Omega + 0.75\Omega = 1.5\Omega$. Hence, using Ohm's Law, the voltage across terminals A and B is $8A \times R_{eq} = 8A \times 1.5\Omega = 12V$.

(b) In this new circuit, what is the Thevenin equivalent voltage across terminals A and B when the switch is open and when the switch is closed?



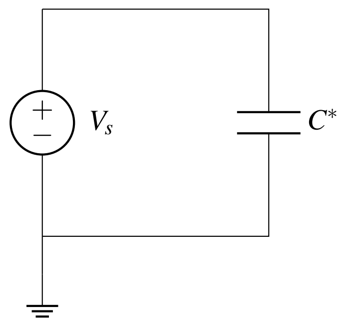
Solution: When the switch is open, the circuit has 2 parallel branches, each with a 1Ω resistor and $8V$ voltage source. The $8A$ from the current source splits equally between both parallel branches. So, $4A$ of current flows through each branch, and hence, the voltage across each resistor is $4A \times 1\Omega = 4V$. The voltage across terminals A and B is the sum of the voltage across a resistor and $8V$ from the voltage source, which is $4V + 8V = 12V$.

When the switch is closed, the top $8V$ voltage source fixes the voltage between the top and center node, and the bottom $8V$ voltage source fixes the voltage between the center node and the bottom node. So the voltage between terminals A and B are the sum of these two voltages which is $8V + 8V = 16V$.

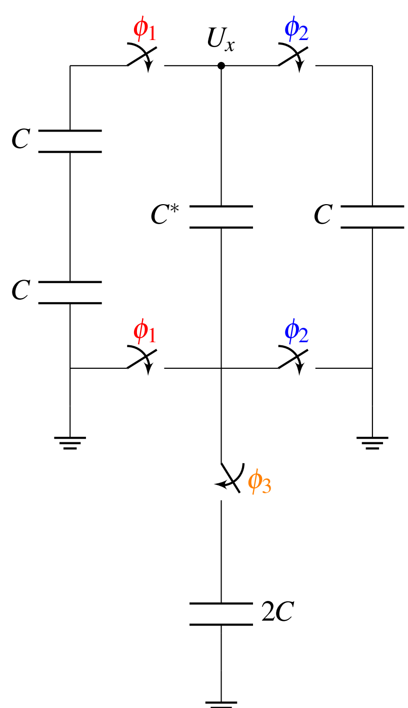
Interesting side note: Notice that part (b) gives us the opposite of our answers in part(a). Part (b) is a circuit example of something called Braess's Paradox, which was first observed in road networks. The paradox is that adding roads to a road network can actually increase traffic! Here, we "add a road" by closing the switch, and the voltage across (i.e., energy per charge to move between) terminals A and B increases!

8. Charge Sharing Choices

(a) C^* is attached to voltage source V_s as shown below and allowed to reach steady state (assume there is zero charge on C^* before it is attached to the voltage source).

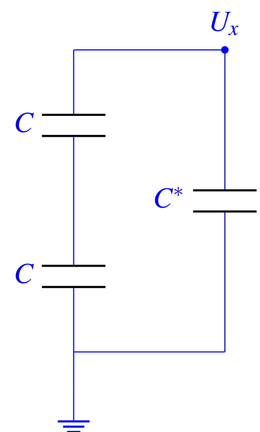


Next, C^* is attached to the circuit shown below. Initially, all the switches are open and the voltage across C^* is still V_s. Which set of switches (ϕ_1, ϕ_2, or ϕ_3), when closed, will cause the potential at node U_x to equal $\frac{1}{2}V_s$ once steady state is reached? Assume that $C^* = C$.

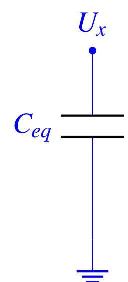


Solution: We can say that the initial charge in the node U_x before switches close is $Q_o = C^*V_s = CV_s$. The easiest way to solve this problem is by inspection. Looking at the figure, we see that when the

ϕ_2 switches close, the charge is evenly distributed between the two equally sized capacitors. Thus, the charge on a single capacitor is $\frac{1}{2}Q_o$, and thus the voltage over this capacitor must also be halved such that $\frac{1}{2}Q_o = C^*\frac{1}{2}V_s = CV_s$. You can also solve this problem by combining capacitors in parallel and series, and then solving for the new voltage across them:
First, let's look at the circuit if the ϕ_1 switches close.



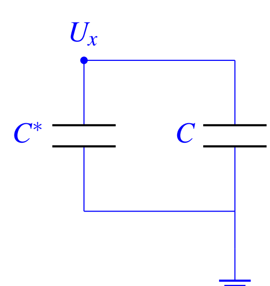
The left two capacitors are in series and can be combined into an equivalent capacitor. This equivalent capacitor is now in parallel with C^* and the circuit looks like the following, where $C_{eq} = \frac{C*C}{C+C} + C = \frac{3}{2}C$.



Now, using charge sharing, the charge on the equivalent capacitor is equal to the charge initially on C^*. Thus:

$$V_sC = Q = U_xC_{eq} = U_x\frac{3}{2}C$$
$$V_s = \frac{3}{2}U_x \neq 2U_x$$

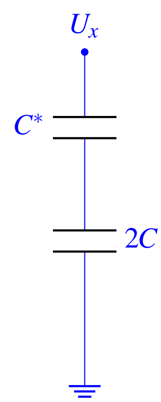
Next, let's look at the circuit if the ϕ_2 switches close:



We see that these two capacitors are in parallel, and thus an equivalent capacitor is $C_{eq} = 2C$.
Using charge sharing:

$$V_s C = Q = U_x C_{eq} = U_x 2C$$
$$V_s = 2U_x$$

And we can see that U_x is half of V_s, as required.
Finally, let's look at the circuit if the circuit if the ϕ_3 switch closes:

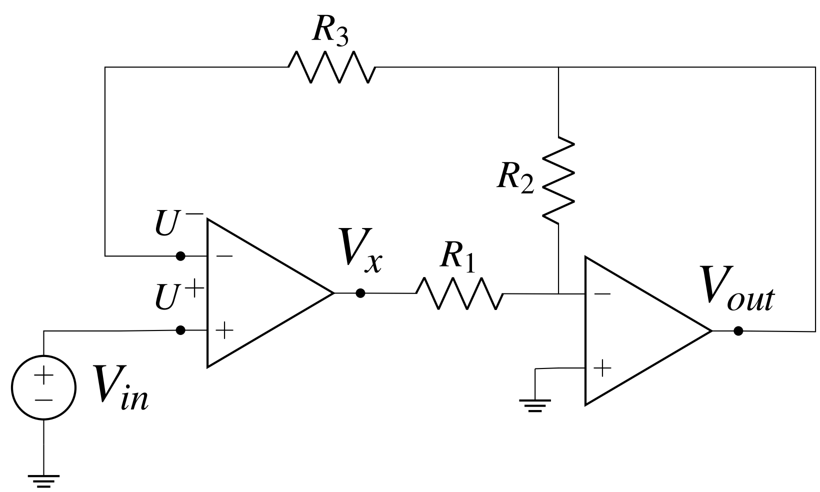


We see that these two capacitors are in series, and thus an equivalent capacitor is $C_{eq} = \frac{C*2C}{C+2C} = \frac{2}{3}C$.
Using charge sharing:

$$V_s C = Q = U_x C_{eq} = U_x \frac{2}{3} C$$
$$V_s = \frac{2}{3} U_x \neq 2U_x$$

9. OpAmps1: Checking for feedback type (6 points)

Version 1:



We are trying to determine if the circuit in the above schematics is in negative or positive feedback. In order to do that, we zero out all independent sources and wiggle the output V_{out} by increasing it. How would the following node voltages change? Note that $V_{error} = U^+ - U^-$

U^-=Increases/Decreases/Stays the same?

U^+=Increases/Decreases/Stays the same?

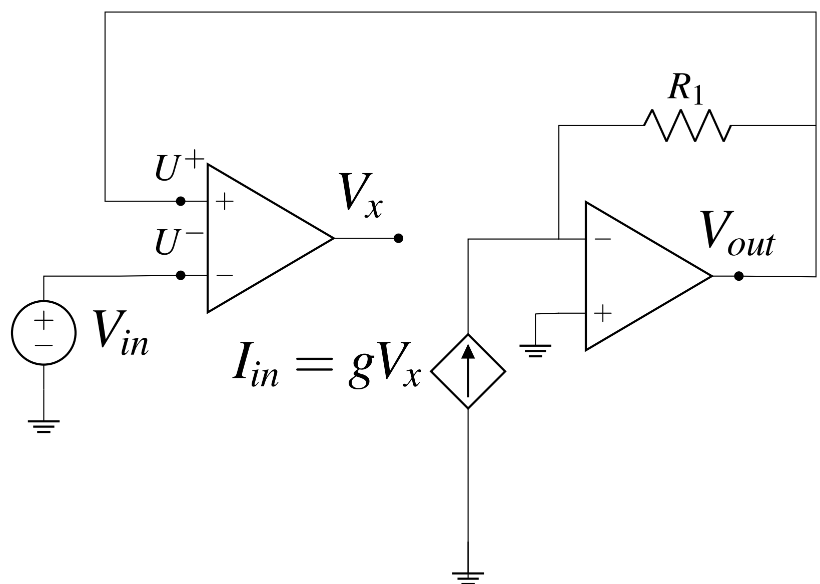
V_{error}=Increases/Decreases/Stays the same?

V_x=Increases/Decreases/Stays the same?

V_{out}=Increases/Decreases/Stays the same?

Based on the change in node voltages, the circuit is in negative/positive feedback.

Version 2:



We are trying to check if the circuit in the above schematics is in negative or positive feedback. In order to do that, we zero out all independent sources and wiggle the output V_{out} by decreasing it. How would the following node voltages and currents change? Note that $V_{error} = U^+ - U^-$ and $I_{in} = gV_x$ where g is a given constant and is equal to $g = 5\text{A/V} = 5/\Omega$

U^-=Increases/Decreases/Stays the same?

U^+=Increases/Decreases/Stays the same?

V_{error}=Increases/Decreases/Stays the same?

V_x=Increases/Decreases/Stays the same?

I_{in}=Increases/Decreases/Stays the same?

V_{out}=Increases/Decreases/Stays the same?

Based on the change in node voltages and currents, the circuit is in negative/positive feedback.

Solution:

Version 1:

U^-: The voltage at the negative terminal of the op amp on the left is equal to $U^- = V_{out} - R_3 I_{R_3}$. $I_{R_3} = I^- = 0A$ which means that the $V_{out} = U^-$ and the increase in V_{out} means that U^- is also **increasing**.

U^+: Zeroing out independent sources means that the voltage source V_{in} becomes a short to gnd. This means that the voltage at U^- would be 0V. The increase in V_{out} would not translate to any change in the potential at node U^+. Hence, the voltage at node U^+ **stays the same**.

V_{error}: Remember that $V_{error} = U^+ - U^-$, U^- increases with increasing V_{out} and while U^+ doesn't change. This means that change in V_{error} is inversely proportional to the change in U^-. Hence, V_{error} decreases with increasing V_{out}.

V_x: This node is the output of op amp. Based on the equivalent model of op amps, the output of op amps can be expressed as $V_x = AV_{error} = A(U^+ - U^-)$. This means that the change in V_x is proportional to the change in V_{error}. Hence, V_x **decreases** with decreasing V_{error}.

V_{out}: Notice that R_1, R_2, and the op amp on the right form an inverting op amp with $V_{out} = V_x \frac{-R_2}{R_1}$. Hence a decreasing V_x means that V_{out} is **increasing**.

Since increasing V_{out} forces the loop to further increase V_{out}, the circuit is in **positive** feedback.

Version 2:

U^-: Since zeroing out all independent sources means that the voltage source V_{in} will be replaced by a wire. This means that the voltage at node U^- would be equal to zero. Hence, decreasing V_{out} will not change the potential at U^- and the voltage will **stay the same.**

U^+: since there is a wire connecting V_{out} to U^+, any changes in U^+ will be proportional to changes in V_{out}. Hence, U^+ will **decrease** with decreasing V_{out}.

V_{error}: Since $V_{error} = U^+ - U^-$ and U^- does not change with V_{out}, the change in V_{error} will be proportional to the change in U^+. Hence, a decrease in U^+ will lead to a **decrease** in V_{error}.

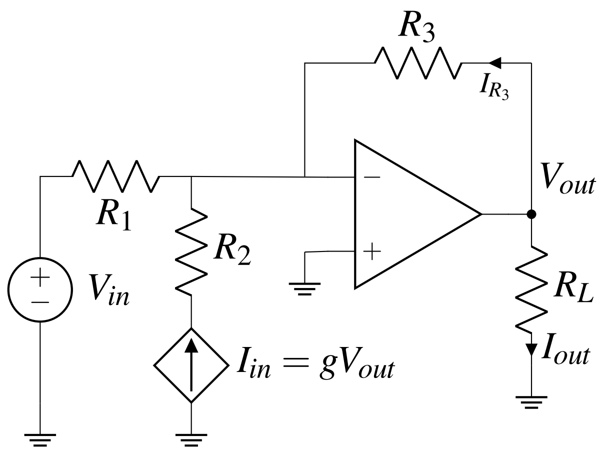
V_x: This node is the output of op amp. Based on the equivalent model of op amps, the output of op amp can be expressed as $V_x = AV_{error} = A(U^+ - U^-)$. This means that the change in V_x is proportional to the change in V_{error}. Hence, V_x **decreases** with decreasing V_{error}.

I_{in}: Since I_{in} is a voltage-controlled current source with $I_{in} = gV_x$. Since g is positive, the change in I_{in} will be proportional to V_x. Hence, a decrease in V_x will lead to a **decrease** in I_{in}.

V_{out}: The op amp on the left, the resistor R_1, and the voltage-controlled current source form a transimpedance amplifier with $V_{out} = -I_{in}R_1$. This means that the changes in I_{in} will be inversely proportional to the changes in I_{in}. Hence, a decrease in I_{in} will lead to an **increase** in V_{out}.

Since the loop corrected the initial change in V_{out}, meaning that an decrease in V_{out} lead to an increase V_{out}, we can see that this circuit is in **negative** feedback.

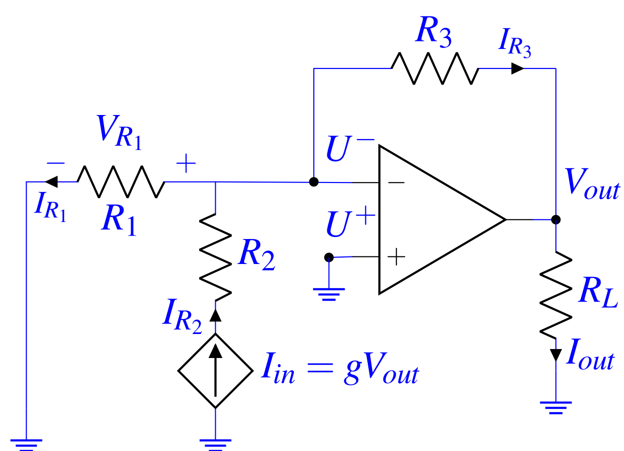
10. Op Amps 2



Based on the above schematics and for $V_{in} = -4\text{V}$, $I_{in} = gV_{out}$, $R_1 = 1\Omega$, $R_2 = 10\text{k}\Omega$, $R_3 = 1\Omega$ and $R_L = 1\text{k}\Omega$, determine if the circuit is in negative feedback, an expression for I_{R_3}, and the value of I_{out}. Note that g is the transconductance is equal to $g = 1\text{A}/\text{V} = 1/\Omega$.

(a) This circuit is negative feedback. True/False?

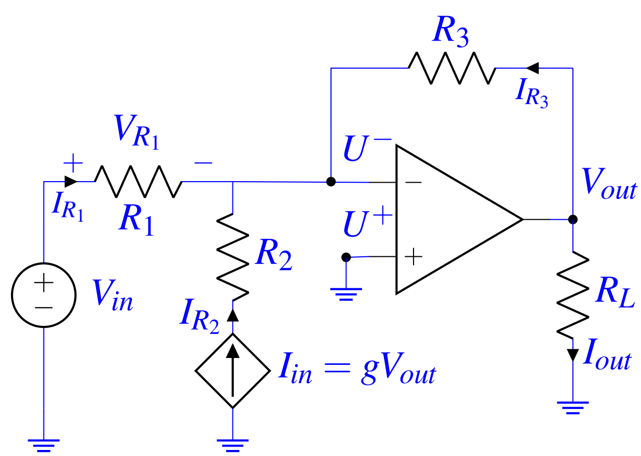
Solution:



To determine whether or not this circuit of is in negative feedback, the first step would be to zero out all independent sources and wiggle the output V_{out} by increasing it. This will cause the voltage at the negative terminal to increase. At the same time, the current through the voltage-controlled current source will increase. By KCL at the negative terminal, we can see that the current I_{in} passes through the resistor R_2 and then it gets divided between R_1 and R_3. The increase in V_{out} leads to an increase in I_{in} and that will lead to an increase in the current through the resistor R_1. We can observe that $V_{R_1} = U^-$ and by Ohm's law $V_{R_1} = I_{R_1}R_1$. Hence, the increase of I_{R_1} means an increase in the voltage V_{R_1} which is the voltage at the negative terminal U^-. This means that overall, the increase in V_{out} leads to an increase in U^- while U^+ remains unchanged. That means that the differential voltage at the input of the op amp, $U^+ - U^-$, will decrease with increasing V_{out}. The decrease in the differential voltage means that the output voltage of the op amp will also decrease since the output is related to the input voltage with the following equation $V_{out} = A(U^+ - U^-)$. Since the initial increase in V_{out} forced the loop to correct that by decreasing V_{out}, this circuit is in **negative feedback**.

(b) What is the correct expression for I_{R_3}?

Solution:



Since this is an op amp circuit in negative feedback, we can write a KCL equation at the terminal of the op amp. From KCL, we know that $I_{R_3} = -I_{R_2} - I_{R_1}$. The current through I_{R_2} is equal to the current through the voltage-controlled current source. This means that $I_{R_2} = I_{in} = gV_{out}$. Assuming this op amp is an ideal op amp with an infinite gain, then we can use the golden rule for op amps Where $U^- = U^+ = 0V$. This means that the current through R_1 is equal to $I_{R_1} = \frac{V_{R_1}}{R_1} = \frac{V_{in}-U^-}{R1} = \frac{V_{in}}{R_1}$. By substituting the values of I_{R_1} and I_{R_2} into the equation to solve for I_{R_3}, we get the following expression for I_{R_3}:

$$I_{R_3} = -(gV_{out} + \frac{V_{in}}{R_1})$$

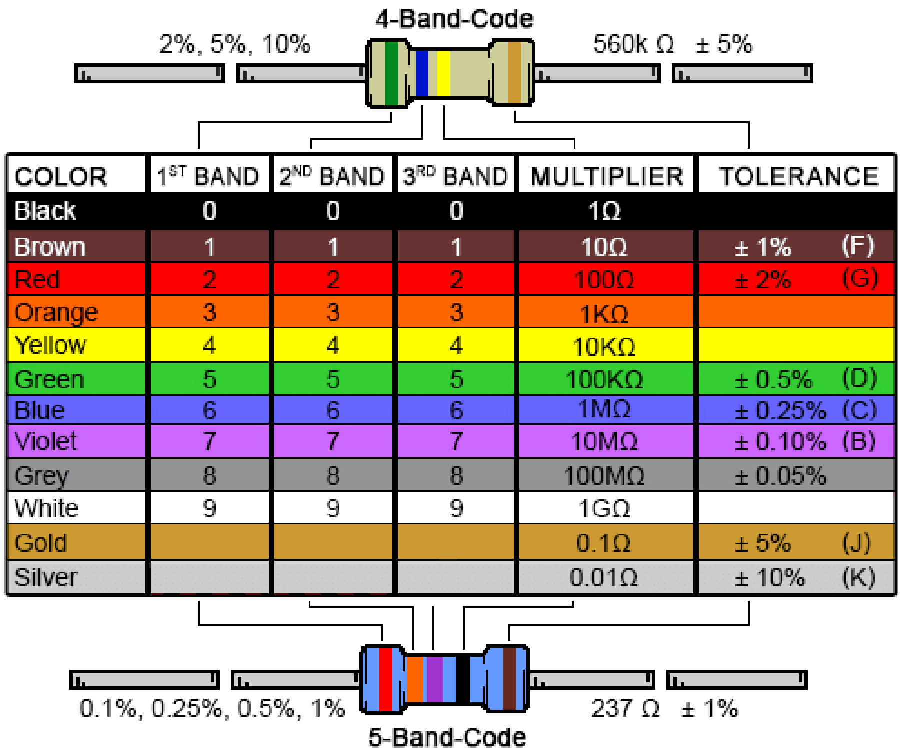
(c) What is I_{out} (mA)?

Solution:

The current I_{out} is equal to the current through the resistor R_L. By Ohm's law, $I_{R_L} = \frac{V_{out}}{R_L}$. Since R_L is given, we just need to calculate the value of V_{out}. Notice that $V_{R_3} = V_{out} - U^- = V_{out}$. And by Ohm's law, $V_{R_3} = I_{R_3}R_3$. It follows that $V_{out} = I_{R_3}R_3$. Since the expression for I_{R_3} is found in the previous part, what's left is isolating V_{out} and plugging in the values for the resistors and the transconductance. For the given values, $V_{out} = 2\text{V}$ and $I_{out} = \frac{V_{out}}{R_L} = \frac{2V}{1k\Omega} = 2\text{mA}$.

11. Resistor Band

(a) Suppose you need a 51Ω resistor your your circuit with a tolerance of ±5%. Referring to the diagram above, which four-band color code would this correspond to?



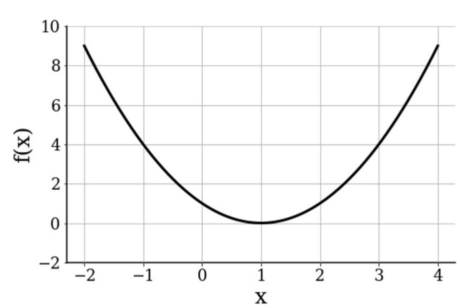
COLOR	1ST BAND	2ND BAND	3RD BAND	MULTIPLIER	TOLERANCE
Black	0	0	0	1Ω	
Brown	1	1	1	10Ω	± 1% (F)
Red	2	2	2	100Ω	± 2% (G)
Orange	3	3	3	1KΩ	
Yellow	4	4	4	10KΩ	
Green	5	5	5	100KΩ	± 0.5% (D)
Blue	6	6	6	1MΩ	± 0.25% (C)
Violet	7	7	7	10MΩ	± 0.10% (B)
Grey	8	8	8	100MΩ	± 0.05%
White	9	9	9	1GΩ	
Gold				0.1Ω	± 5% (J)
Silver				0.01Ω	± 10% (K)

Solution: We need to use the first two bands to pick the correct numbers and then choose the correct multipler. We can choose 5-1 with a multiplier of 1Ω to achieve the desired resistance. Finally, we need to choose the correct fourth band to get a tolerance of $\pm 5\%$. This corresponds to green-brown-black-gold.

12. Orthogonality

(a) $f(x)$ and $g(x)$ are both polynomials with degree at most 2. $f(x) = x^2 - 2x + 1$ ($f(x) = -x^2 + 2x - 1$ for the alternate version). We define inner product between two polynomials as $\langle f(x), g(x) \rangle = f(0)g(0) + f(1)g(1) + f(2)g(2)$.

If $g(x)$ is orthogonal to $f(x)$, which is possible equation for $g(x)$?



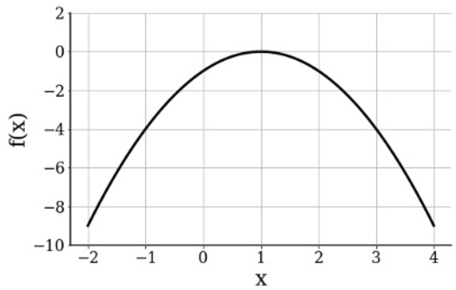
i. $-x^2 + 2x - 1$

ii. $x^2 + x - 1$

iii. $x - 1$

iv. x

Alternate version:



i. $2x^2 - 5x - 2$
ii. $5x + 2$
iii. $-x^2 + 2x - 1$
iv. $-x + 1$

Solution: Orthogonality means the inner product is zero, namely $\langle f(x), g(x) \rangle = 0$.

i. $\langle f(x), g(x) \rangle = 1 \times (-1) + 0 \times 0 + 1 \times (-1) = -2$.
ii. $\langle f(x), g(x) \rangle = 1 \times (-1) + 0 \times 1 + 1 \times 5 = 4$.
iii. $\langle f(x), g(x) \rangle = 1 \times (-1) + 0 \times 0 + 1 \times 1 = 0$. $g(x)$ is orthogonal to $f(x)$.
iv. $\langle f(x), g(x) \rangle = 1 \times 0 + 0 \times 1 + 1 \times 2 = 2$.

Using a similar procedure, $g(x) = -x + 1$ for the alternate version.

13. Satellite Codes

(a) Which of the following vectors is best suited as a satellite code?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Solution: A satellite code should have a very small autocorrelation at any time shift other than 0.

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ has an autocorrelation at all times of $< \vec{v}_1, \vec{v}_1 >= 1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 1 = 5$.

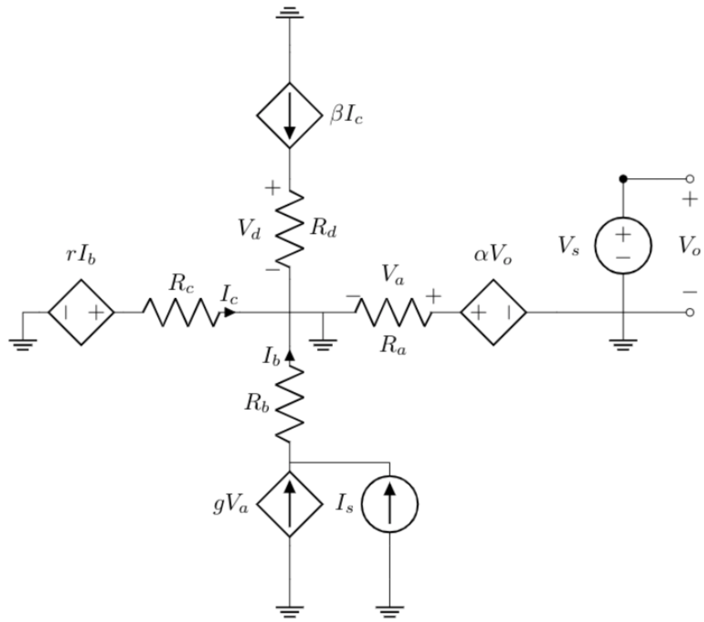
This means that there is not any discrimation between time shifts. This property makes this vector a poor code.

$< \vec{v}_2, \vec{v}_2 >= 1 \times 1 + 1 \times 1 + 1 \times 1 + (-1) \times (-1) + 1 \times 1 = 5$ at shift 0, but yields 1 at all other shifts, which makes this vector a good code.

$< \vec{v}_3, \vec{v}_3 >= 1 \times 1 + (-1) \times (-1) + 1 \times 1 + (-1) \times (-1) + 1 \times 1 = 5$ at shift 0, but yields -3 at time shifts 1 and 4, which makes this vector a poor code.

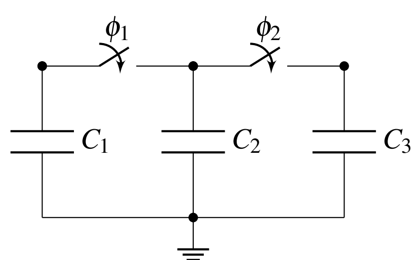
14. Superposition Fun

(a) Suppose we want to solve this circuit using superposition. If we want to start solving by leaving V_s on, how many other sources do we need to null out?



Solution: One. This circuit only has two independent sources, so if we leave V_s on, we only need to null out I_s

15. Charge Sharing Cycles



The circuit begins in phase 1 (all switches open). During phase 1, the ϕ_1 switch is closed and the ϕ_2 switch is open. During phase 2, the ϕ_2 switch is closed and the ϕ_1 switch is open. The circuit moves from phase 0 to phase 1 to phase 2 and then back to phase 0 (defined as a cycle). Let t indicate the number of cycles completed and $\vec{x}[t] = \begin{bmatrix} Q_{C_1}^{(t)} \ Q_{C_2}^{(t)} \ Q_{C_3}^{(t)} \end{bmatrix}^T$ indicate the distribution of charge across the three capacitors after t cycles have occurred. Let $\vec{x}[0] = \begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}^T$. You are given $C_1 = C_2 = C_3$.

(a) Find $\vec{x}[1]$, the distribution of charges after phase 1 and phase 2 have been completed once.

Solution:

ϕ_0 to ϕ_1 transition:

When the ϕ_1 switch closes, the initial 1 Coulomb on C_1 splits evenly across C_1 and C_2. Thus C_1 and

C_2 will each have $\frac{1}{2}$ Coulombs of charge. This transition can be modeled by $T_{\phi_1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

ϕ_1 to ϕ_2 transition:

When the ϕ_1 switch opens and the ϕ_2 switch closes, the $\frac{1}{2}$ Coulomb on C_2, splits evenly across C_2 and C_3. Thus C_2 and C_3 will each have $1/4$ Coulomb of charge. This transition can be modeled by $T_{\phi_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

ϕ_2 to ϕ_0 transition:

When ϕ_2 opens, the distribution of charge won't change.

Result

Therefore $\vec{x}[1] = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$

(b) Find a matrix T such that $T\vec{x}[t] = \vec{x}[t+1]$.

Solution: The transition for a complete cycle is represented by the transition from ϕ_0 to ϕ_1 and from ϕ_1 to ϕ_2.

Thus $T = T_{\phi_2} T_{\phi_1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

(c) Find $\lim_{t\to\infty} \vec{x}[t]$.

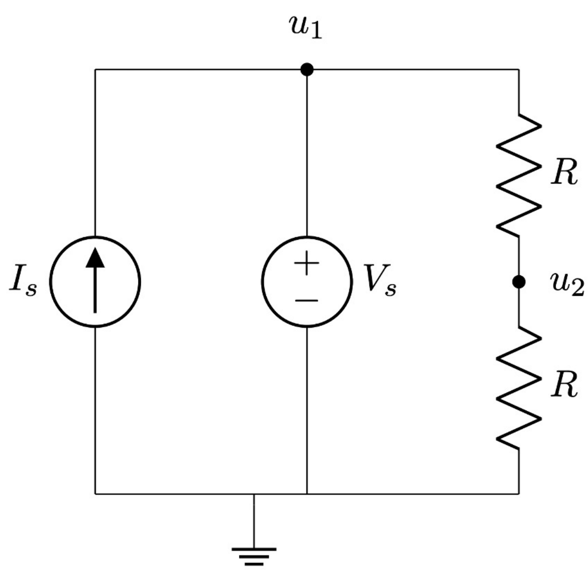
Solution: After the ϕ_0 to ϕ_1 transition, the charge is distributed equally across C_1 and C_2. After the ϕ_1 to ϕ_2 transition, the charge is distributed equally across C_2 and C_3. Intuitively this means that the charge will be uniformly distributed across the three capacitors after many cycles. Therefore $\lim_{t\to\infty} \vec{x}[t] = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$

Alternate solution:

Note that T is a conservative transition matrix, thus an eigenvalue of $\lambda = 1$ exists. Therefore the initial distribution will converge to the steady state defined by the eigenvector associated with $\lambda = 1$: $\vec{v}_{\lambda=1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$

16. Linear Algebra Circuit (22 points)

(a) Consider the following circuit (4 points):



Let $\vec{x} = [V_s \;\; I_s]^T$, and let $\vec{u} = [u_1 \;\; u_2]^T$. Construct A such that $A\vec{x} = \vec{u}$, choose the most correct statement: For a given $\vec{u} = \vec{u}_0$, the matrix equation $A\vec{x} = \vec{u}_0$:

Options:

1) can have a single solution or no solution;

2) always has a single solution;

3) always has no solution;

4) always has infinite solutions;

5) can have infinite solutions or no solutions;

Solution: As shown in the circuit above, u_1 is directly connected to the voltage source V_s, then, we will have $u_1 = V_s$, and $u_2 = \frac{V_s}{2}$ as a voltage divider. Therefore, we can formulate the matrix equation as:

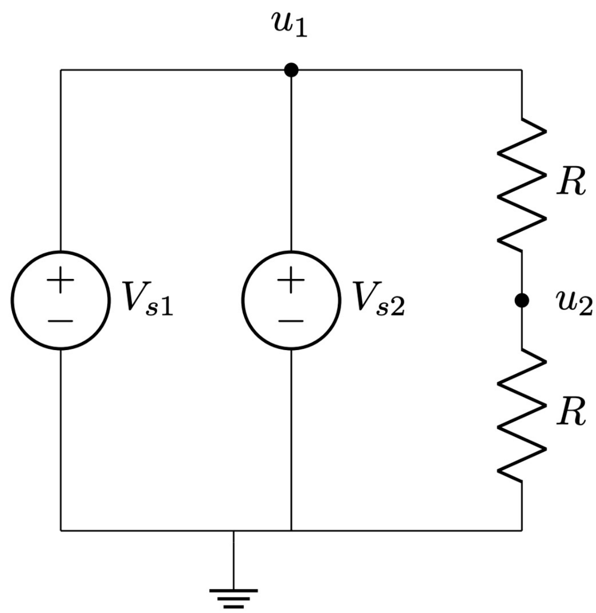
$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Solving it by Gaussian Elimination, we have:

$$\left(\begin{array}{cc|c} 1 & 0 & u_1 \\ \frac{1}{2} & 0 & u_2 \end{array} \right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 0 & u_1 \\ 0 & 0 & u_2 - \frac{u_1}{2} \end{array} \right).$$

If $u_1 = 2u_2$, the equation has infinite solutions, otherwise if $u_1 \neq 2u_2$, there will be no solutions. 5) is the correct answer.

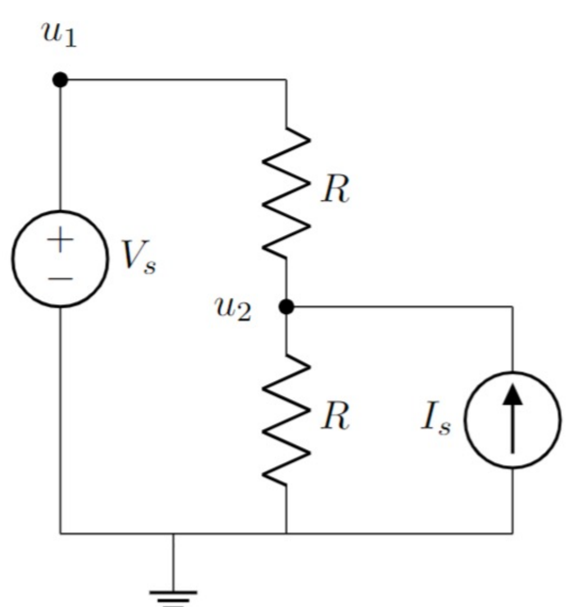
(b) Consider the following circuit (2 points):



Let $\vec{x} = [V_{s1} \;\; V_{s2}]^T$, and let $\vec{u} = [u_1 \;\; u_2]^T$. Construct A such that $A\vec{x} = \vec{u}$. For a given $\vec{u} = \vec{u}_0$, the matrix equation $A\vec{x} = \vec{u}_0$ has a solution for the vector $\vec{x}$, then the solution is in the span of $[a \;\; 0.5]$, where a = ?.

Solution: Note that in the circuit above, two voltage sources V_{s1} and V_{s2} are in parallel. For an ideal circuit, this can only happens when $V_{s1} = V_{s2}$. Therefore, the solution is in the span of $[0.5 \;\; 0.5]^T$, and $a = 0.5$.

(c) Consider the following circuit (6 points):



Let $\vec{x} = [V_s \;\; I_s]^T$, and let $\vec{u} = [u_1 \;\; u_2]^T$. Let $\vec{u} = \vec{u}_0$ and construct A such that $A\vec{x} = \vec{u}_0$. Choose the correct statements about:

1) A:

i. Matrix A is invertible;
ii. Matrix A is not invertible;
iii. Not possible to determine if A is invertible.

2) u_1:

i. u_1 depends only on V_s;
ii. u_1 depends only on I_s;
iii. Neither of the other choices;

3) u_2:

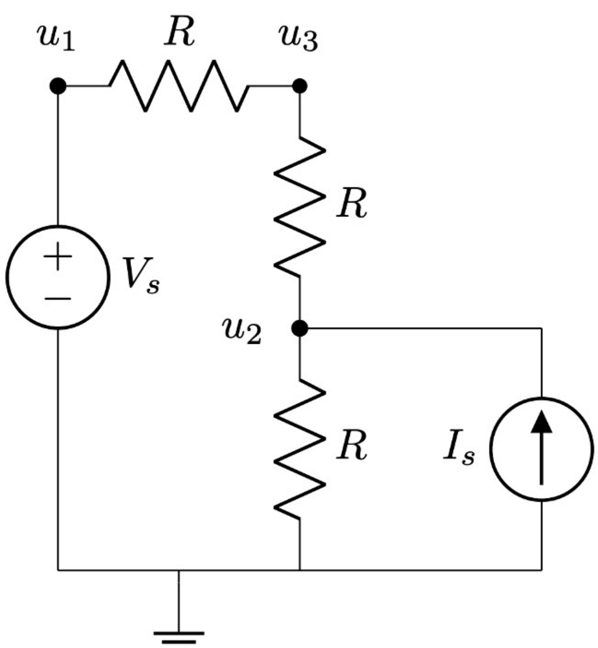
i. u_2 depends only on V_s;
ii. u_2 depends only on I_s;
iii. Neither of the other choices;

Solution: Here, we use superposition to solve this circuit. Step 1: zero out I_s (make it open circuit), we have $u_{1,V_s} = V_s$ and $u_{2,V_s} = \frac{V_s}{2}$ as a voltage divider. Step 2: zero out V_s (short it), we have $u_{1,I_s} = 0$ and $u_{2,I_s} = \frac{I_s R}{2}$. Using superposition, we have $u_1 = u_{1,V_s} + u_{1,V_s} = V_s$ and $u_2 = u_{2,V_s} + u_{2,V_s} = \frac{V_s + I_s R}{2}$. Therefore, we can formulate the matrix equation as:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{R}{2} \end{bmatrix} \begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $\det(A) = \frac{R}{2} \neq 0$, A is invertible. u_1 depends only on V_s and u_2 depends on both V_s and I_s. The correct answer will then be 1)i, 2)i, 3)iii.

(d) Consider the following circuit (10 points):



Let $\vec{x} = [V_s \;\; I_s]^T$, and let $\vec{u} = [u_1 \;\; u_2 \;\; u_3]^T$. Assume that all measurements of the node voltages are noisy with some small error. Given $R = 1\Omega$, $\vec{u} = \vec{u}_0 = [3 \;\; 2 \;\; 2]^T$. We can construct $A = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ such that $A\vec{x} = \vec{u}_0$.

What is the best approximation for V_s and I_s in volts and amperes, respectively?

Solution: Here, we also show how to construct matrix A. We use superposition to solve this circuit. Step 1: zero out I_s (make it open circuit), we have $u_1 = V_s$, $u_2 = \frac{V_s}{3}$, and $u_3 = \frac{2V_s}{3}$. Step 2: zero out the voltage source (short it), we have $u_1 = 0$, $u_2 = \frac{2I_sR}{3}$, and $u_3 = \frac{u_1+u_2}{2} = \frac{I_sR}{3}$. Using superposition, we have $u_1 = u_{1,V_s} + u_{1,I_s} = V_s$, $u_2 = u_{2,V_s} + u_{2,I_s} = \frac{V_s+2I_sR}{3}$, and $u_3 = u_{3,V_s} + u_{3,I_s} = \frac{2V_s+I_sR}{3}$. Given $R = 1\Omega$, we can formulate the matrix equation as:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix},$$

where $A = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, and $A\vec{x} = \vec{u}_0$. Since we have more equations then unknows, we use Least Square to solve it and we have:

$$\vec{x} = (A^TA)^{-1}A^T\vec{u}_0 = \begin{bmatrix} \frac{17}{6} \\ \frac{4}{3} \end{bmatrix}.$$

Some intermediate results: $A^TA = \begin{bmatrix} \frac{14}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix}$, $(A^TA)^{-1} = \begin{bmatrix} \frac{5}{6} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{bmatrix}$, $A^T\vec{u}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $(A^TA)^{-1}A^T\vec{u}_0 = \begin{bmatrix} \frac{17}{6} \\ \frac{4}{3} \end{bmatrix}$. Therefore, we have $V_s = \frac{17}{6}$ and $I_s = \frac{4}{3}$.