EECS 16A Designing Information Devices and Systems I
Fall 2022 Midterm 1

Midterm 1 Solution

1. **HONOR CODE**
   Please read the following statements of the honor code, and sign your name (you don't need to copy it).

   I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

   - *I have read the instructions for this exam. I understand them and will follow them.*
   - *All of the work submitted here is my original work.*
   - *I did not reference any sources other than my unlimited printed resources.*
   - *I did not collaborate with any other human being on this exam.*

2. **Tell us about something that makes you happy (1 point)** *All answers will be awarded full credit.*

3. Oski on Sproul (12 points)

Oski is studying how long it takes students to get through Sproul plaza. In particular, he is interested in how long each interaction takes in seconds: talking to a club *(t)*, accepting a flyer *(a)*, rejecting a flyer *(r)*, and taking a picture *(p)*.

- Ayah talked to 2 clubs, accepted 10 flyers, rejected 4 flyers, and took 1 picture. Her trip took a total of 96 seconds.
- Mira talked to 0 clubs, accepted 6 flyers, rejected 15 flyers, and took 2 pictures. Her trip took a total of 171 seconds.
- Rishi talked to 1 club, accepted 2 flyers, rejected 14 flyers, and took 2 pictures. Her trip took a total of 58 seconds.

(a) (4 points) Formulate a matrix-vector representation for this set of linear equations in the form of $\mathbf{A}\vec{x} = \vec{b}$ where *t, a, r, p* are the variables in $\vec{x}$.

Solution:

$$\mathbf{A} = \begin{bmatrix} 2 & 10 & 4 & 1 \\ 0 & 6 & 15 & 2 \\ 1 & 2 & 14 & 2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} t \\ a \\ r \\ p \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 96 \\ 171 \\ 58 \end{bmatrix}$$

(b) (4 point) Oski's collaborator, Tree, gathered some more data and arrived at matrix $\mathbf{A}$ and vector $\vec{b}$ below. However, Tree wrote in cursive and Oski cannot read the value represented by $\mathbf{u}$ in matrix $\mathbf{A}$. What values of $\mathbf{u}$ would guarantee *no solution* to $\mathbf{A}\vec{x} = \vec{b}$? Justify your answer.

$$\mathbf{A} = \begin{bmatrix} 2 & 17 & 0 & 3 \\ 8 & 2 & 0 & 1 \\ 16 & 4 & \mathbf{u} & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 70 \\ 30 \\ 60 \\ 90 \end{bmatrix}$$

Solution: We can write the augmented matrix:

$$\mathbf{M} = \left[\begin{array}{cccc|c} 2 & 17 & 0 & 3 & 70 \\ 8 & 2 & 0 & 1 & 30 \\ 16 & 4 & \mathbf{u} & 2 & 60 \\ 0 & 0 & 2 & 0 & 90 \end{array}\right]$$

u must not be 0. We see that the third row is a scaled version of the second row. If u were not equal to 0, these two rows would contradict and yield no solution.

(c) (4 point) Finally, Oski asks you to help out with one more augmented matrix problem. Help solve it using Gaussian elimination.

$$\mathbf{M} = \left[\begin{array}{ccc|c} 4 & 4 & 0 & 24 \\ 1 & 3 & 0 & 14 \\ 2 & 2 & 6 & 18 \end{array}\right]$$

Solution:

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 24 \\ 1 & 3 & 0 & 14 \\ 2 & 2 & 6 & 18 \end{array}\right] \xrightarrow[\frac{1}{2}R_3 \to R_3]{\frac{1}{4}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 1 & 3 & 0 & 14 \\ 1 & 1 & 3 & 9 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 3 & 3 \end{array}\right] \xrightarrow{2R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

From here we get the final solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

4. Matrix Multiverse (20 points)

For the following questions, **circle one option that most accurately completes the statement**.
Then provide a **brief explanation** of your choice.

(a) (5 points) Consider a set of n linearly independent vectors $\{\vec{w}_1, \ldots, \vec{w}_n\} \in \mathbb{R}^n$. A vector $\vec{u} \in \mathbb{R}^n$ will:

Option 1. Always be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$
Option 2. Sometimes be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$
Option 3. Never be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$

Explain your choice:

Solution: Option 1. The span of n linearly independent vectors $\vec{w}_1, \ldots, \vec{w}_n$ is $\mathbb{R}^n$. In other words, $span\{\vec{w}_1, \ldots, \vec{w}_n\} = \mathbb{R}^n$. Consequently, any vector $\vec{u} \in \mathbb{R}^n$ will be a linear combination of the set $\{\vec{w}_1, \ldots, \vec{w}_n\}$.

(b) (5 points) Consider a randomly chosen set of n vectors $\{\vec{z}_1, \ldots, \vec{z}_n\} \in \mathbb{R}^n$. The matrix whose columns are formed by all the vectors in this set will:

Option 1. Always be invertible
Option 2. Sometimes be invertible
Option 3. Never be invertible

Explain your choice:

Solution: Option 2. Since we are not told whether the given vectors are linearly independent or not, we cannot say the matrix will always or never be invertible. The matrix will be invertible if the set is a linearly independent set of vectors, otherwise if the set of vectors is linearly dependent, the matrix will not be invertible. The most accurate choice is option 2.

(c) (5 points) Consider the following augmented matrix where * can be any value that is not 0 and not 1:

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 1 & * & * \end{array}\right]$$

The system of linear equations represented:

Option 1. has a unique solution
Option 2. has no unique solution
Option 3. has infinite solutions
Option 4. has no solutions

Explain your choice:

Solution: Option 2. Since there is a missing pivot in the second column, it is absolute that there is no unique solution. Option 3 and 4 are also possible from the information given. However, because the matrix is not fully row reduced, we cannot guarantee that there are infinite solutions (meaning rows are scalar multiples of each other) or guarantee that there are no solutions (rows are inconsistent). The question asks to choose the most accurate statement. Thus, option 2 is the correct answer because only option 2 is absolutely true.

(d) (5 points) Finally, consider a matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$. If $\mathbf{B}$ has a non-zero determinant, then:

Option 1. it must have linearly dependent columns
Option 2. the columns of $\mathbf{B}$ span $\mathbb{R}^n$
Option 3. it must have a non-trivial null space
Option 4. one solution to the characteristic polynomial must be zero

Explain your choice:

Solution: Option 2. We are told the matrix has a non-zero determinant, which means the matrix is invertible, and the columns of $\mathbf{B}$ are linearly independent. Thus, the columns of $\mathbf{B}$ must span $\mathbb{R}^n$. Furthermore, because the matrix is invertible, there must be a trivial null space, has only non-zero eigenvalues, and thus no solution to the characteristic polynomial will be zero. The only correct statement left is Option 2.

5. Matrix Madness (18 points)

For the following subparts, consider the matrix $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) (5 points) What is the value of a that satisfies the expression below?

$$Null(\mathbf{A}) = \text{span}\left(\begin{bmatrix} a \\ 2 \end{bmatrix}\right)$$

Solution: The column vectors of $\mathbf{A}$ are linearly dependent as one is a scalar multiple of the other. We can show this more formally by solving the system using Gaussian elimination when $\mathbf{A}\vec{u} = \vec{0}$

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Expressing the vector $\vec{u}$ in terms of the free variables yields

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_2 \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Finally, the nullspace of $\mathbf{A}$ can be written as

$$Null(\mathbf{A}) = \text{span}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

Thus, a = -1. Full credit can be given for correctly identifying a = -1 by inspection.

(b) (5 points) What is the value of b that satisfies the new expression below?

$$Col(\mathbf{A}) = \text{span}\left(\begin{bmatrix} b \\ 1 \end{bmatrix}\right)$$

Solution: The column vectors of $\mathbf{A}$ are linearly dependent as one is a scalar multiple of the other. Inspection can be used to determine the span as a scalar multiple (based on the partial solution provided) of either column of $\mathbf{A}$.

$$Col(\mathbf{A}) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

This problem can also be solved using Gaussian elimination to parametrically identify the columnspace (i.e., span of the columns) of $\mathbf{A}$.

$$\left[\begin{array}{cc|c} 4 & 2 & u_1 \\ 2 & 1 & u_2 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 2 & 1 & \frac{1}{2}u_1 \\ 2 & 1 & u_2 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 2 & 1 & \frac{u_1}{2} \\ 0 & 0 & u_2 - \frac{1}{2}u_1 \end{array}\right]$$

The solution $\vec{b}$ is only consistent if the second row (i.e., equation) is satisfied. In other words,

$$u_2 = \frac{1}{2}u_1 \quad \implies \quad \vec{u} = u_1 \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Finally, the columnspace of $\mathbf{A}$ can be written as

$$Col(\mathbf{A}) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

Thus, b = 2. Full credit can be given for correctly identifying b = 2 by inspection.

(c) (8 points) An arbitrary matrix $\mathbf{B}$ satisfies the following equations:

$$\mathbf{B}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}6\\-3\end{bmatrix}$$

$$\mathbf{B}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

What is $\mathbf{B}\begin{bmatrix}5\\-2\end{bmatrix}$?

Solution: There are two methods to solve this problem.

Method 1: We can deduce the eigenvalues and eigenvectors from the information given about $\mathbf{B}$.
From the first line, we can see that $\begin{bmatrix}6\\-3\end{bmatrix}=3\begin{bmatrix}2\\-1\end{bmatrix}$. Thus, $\lambda_1=3$ and $\vec{v}_1=\begin{bmatrix}2\\-1\end{bmatrix}$.
From the second line, we can again use the same method to write $\begin{bmatrix}1\\0\end{bmatrix}=0\begin{bmatrix}0\\0\end{bmatrix}$. Thus, $\lambda_2=0$ and $\vec{v}_2=\begin{bmatrix}1\\0\end{bmatrix}$.
Finally, recall that we can solve any matrix-vector multiplication $\mathbf{B}\vec{x}$ by decomposing $\vec{x}$ into the eigenvectors of $\mathbf{B}$, $\vec{v}_1$ and $\vec{v}_2$.

$$\begin{bmatrix}5\\-2\end{bmatrix}=2\begin{bmatrix}2\\-1\end{bmatrix}+1\begin{bmatrix}1\\0\end{bmatrix}=2\vec{v}_1+1\vec{v}_2$$
$$\mathbf{B}\begin{bmatrix}5\\-2\end{bmatrix}=\mathbf{B}(2\vec{v}_1+1\vec{v}_2)=2\lambda_1\vec{v}_1+\lambda_2\vec{v}_2$$

Now we can plug in the eigenvectors and eigenvalues to solve the question.

$$2\lambda_1\vec{v}_1+\lambda_2\vec{v}_2=2(3)\begin{bmatrix}2\\-1\end{bmatrix}+0\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}12\\-6\end{bmatrix}$$

Method 2: We can use the information given to solve for matrix $\mathbf{B}$. Let's represent the values of $\mathbf{B}$ with the variables b_i such that $\mathbf{B}=\begin{bmatrix}b_1 & b_2\\b_3 & b_4\end{bmatrix}$.
We can write a system of linear equations from the information given:

$$2b_1-b_2=6$$
$$2b_3-b_4=-3$$
$$b_1+0b_2=0$$
$$b3+0b_4=0$$

From the last two equations, we get $b_1=0, b_3=0$. We can back substitute and find $b_2=-6, b_4=3$.
Thus, $\mathbf{B}=\begin{bmatrix}0 & -6\\0 & 3\end{bmatrix}$, and finally $\mathbf{B}\begin{bmatrix}5\\-2\end{bmatrix}=\begin{bmatrix}12\\-6\end{bmatrix}$.

6. Hungry for Linearity (16 points)

(a) Determine whether the following functions are linear. If so, show they are linear with the properties of linearity. If the function is not linear, clearly demonstrate at least one property of linearity is violated.

Solution: Linearity can be proven using homogeneity and superposition simultaneously (as presented here), or with the two properties written out separately. To prove the function is not linear, only one property of linearity needs to be shown for credit.

i. (4 point) $g(x_1, x_2, x_3) = -\pi x_1 + e^6 x_2 - \sqrt{2} x_3$

Solution: The function is linear. We can test homogeneity and superposition:

$$\begin{aligned} &g(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) \\ &= -\pi(\alpha a_1 + \beta b_1) + e^6(\alpha a_2 + \beta b_2) - \sqrt{2}(\alpha a_3 + \beta b_3) \\ &= -\pi \alpha a_1 + e^6 \alpha a_2 - \sqrt{2}\alpha a_3 - \pi \beta b_1 + e^6 \beta b_2 - \sqrt{2}\beta b_3 \\ &= \alpha g(a_1, a_2, a_3) + \beta g(b_1, b_2, b_3) \end{aligned}$$

ii. (4 points) $f(x) = 3\sqrt{x^2}$

Solution: This function is not linear. It violates both homogeneity and superposition. Either solution is accepted. First, let's look at homogeneity:

$$f(\alpha x) = 3\sqrt{\alpha^2 x^2}$$

Take the case where $\alpha = -1$. To satisfy homogeneity, $f(-x) = -f(x)$. However, because this function always produces a positive value ($f(-x) > 0$) this function violates homogeneity.
Now let's look at superposition:

$$f(x_1 + x_2) = 3\sqrt{(x_1 + x_2)^2}$$

To satisfy superposition, $f(x_1 + x_2) = f(x_1) + f(x_2) = 3\sqrt{x_1{}^2} + 3\sqrt{x_2{}^2}$. We see from the equation above, this is not the case. The function violates superposition.

(b) Now consider an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and vector $\vec{x} \in \mathbb{R}^{n \times 1}$. Determine whether the following functions are linear. If so, show they are linear with the properties of linearity. If the function is not linear, clearly demonstrate at least one property of linearity is violated.

i. (4 points) $f(\vec{x}) = \mathbf{A}^2 \vec{x}$

Solution: The function is linear. We can test homogeneity and superposition:

$$\begin{aligned} &f(\alpha \vec{x} + \beta \vec{y}) \\ &= \mathbf{A}^2(\alpha \vec{x} + \beta \vec{y}) \\ &= \alpha \mathbf{A}^2 \vec{x} + \beta \mathbf{A}^2 \vec{y} \\ &= \alpha f(\vec{x}) + \beta f(\vec{y}) \end{aligned}$$

ii. (4 points) $f(\vec{x}) = \mathbf{A}\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and vector $\vec{x} \in \mathbb{R}^{2\times 1}$.

Solution: The function is nonlinear because it violates both homogeneity and superposition. This would be the case for any vector added where $\vec{c} \neq \vec{0}$. In this case, $\vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but we can generalize. First, let's look at homogeneity:

$$f(\alpha\vec{x}) = \mathbf{A}\alpha\vec{x} + \vec{c}$$

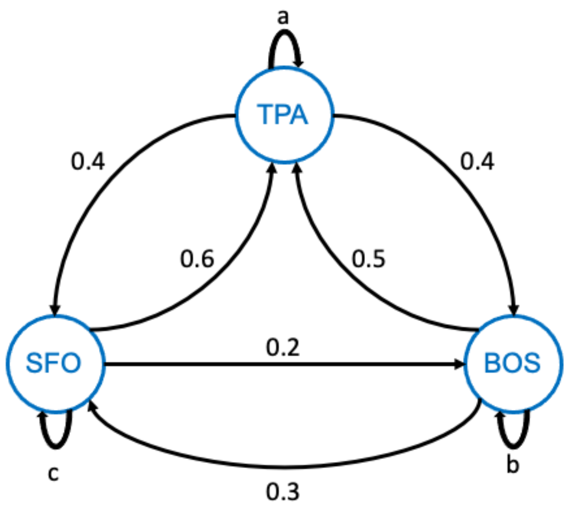
In order to satisfy homogeneity, $f(\alpha\vec{x}) = \alpha f(\vec{x}) = \alpha(\mathbf{A}\vec{x} + \vec{c})$. However, we can see from above that this is not the case. This function violates homogeneity. Now let's take a look at superposition:

$$f(\vec{x}_1) + f(\vec{x}_2) = \mathbf{A}\vec{x}_1 + \vec{c} + \mathbf{A}\vec{x}_2 + \vec{c} = \mathbf{A}(\vec{x}_1 + \vec{x}_2) + 2\vec{c}$$

In order to satisfy superposition, $f(\vec{x}_1) + f(\vec{x}_2) = f(\vec{x}_1 + \vec{x}_2) = \mathbf{A}(\vec{x}_1 + \vec{x}_2) + \vec{c}$. Again, we see that this is not the case and the function violates superposition.

7. Flight Tracking (20 points)

Your friend decides to describe the air traffic through three airports in the following graph:



(a) (6 point) Let $\vec{p}[t] = \begin{bmatrix} p_t[t] \\ p_b[t] \\ p_s[t] \end{bmatrix}$ where $p_t[t], p_b[t], p_s[t]$ represent the number of airplanes at TPA, BOS, and SFO at time t respectively.

Determine $\mathbf{A}$ such that $\vec{p}[t+1] = \mathbf{A}\vec{p}[t]$. What values of **a**, **b** and **c** would make the system conservative?

Solution: Reading out the diagram,

$$\mathbf{A} = \begin{bmatrix} T \to T & B \to T & S \to T \\ T \to B & B \to B & S \to B \\ T \to S & B \to S & S \to S \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a & 0.5 & 0.6 \\ 0.4 & b & 0.2 \\ 0.4 & 0.3 & c \end{bmatrix}$$

For the system to be conservative:

$a + 0.4 + 0.4 = 1 \implies a = 0.2$

$0.5 + b + 0.3 = 1 \implies b = 0.2$

$0.6 + 0.2 + c = 1 \implies c = 0.2$

You point out that if a, b, or c have a value greater than 0, this means an airplane departing an airport arrives back at the same airport. You friend comes back with the following new transition matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

(b) (6 points) Your friend says $\vec{p}[n]$ is $\begin{pmatrix} 300 \\ 200 \\ 400 \end{pmatrix}$. Is it possible to determine the state vector at the previous timestep $n-1$? Justify why or why not.

Solution: In order to determine the state vector at the previous time step, $\mathbf{B}^{-1}$ must exist. To check the invertibility of $\mathbf{B}$, we need to check that the columns of $\mathbf{B}$ are linearly independent. We can use Gaussian Elimination. Let's multiply the entire matrix by 2 to simplify our calculations:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_1} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{2}R_3]{R_1 + \frac{1}{2}R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since there is a pivot in each column of the matrix, we know the column vectors are linearly independent and the matrix $\mathbf{B}^{-1}$ exists. Thus, we are able to determine the state vector at the previous time step.

(c) (8 points) Your friend tells you that the eigenvalues of $\mathbf{B}$ are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$, and $\lambda_3 = -\frac{1}{2}$. Given that $\vec{p}[0] = \begin{pmatrix} 300 \\ 220 \\ 380 \end{pmatrix}$, what is the number of airplanes at each airport after infinite timesteps?

$$\mathbf{B} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

Solution: Since every is less than or equal to one, we know that all initial states will converge to a steady-state after infinite timesteps. We know the steady-state solution corresponds to the eigenvalue of 1, thus we need to solve for the vector $\vec{v}$ such that $(\mathbf{B} - \mathbf{I})\vec{v} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -1 & 0.5 & 0.5 & 0 \\ 0.5 & -1 & 0.5 & 0 \\ 0.5 & 0.5 & -1 & 0 \end{array}\right] \xrightarrow{\times 2} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_2]{2R_3 + R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$
$$\xrightarrow[R_1 - R_2 \to R_2]{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

From here, we can write the following:

$$x_2 - x_3 = 0$$
$$x_1 - x_3 = 0$$

Letting x_3 be a free variable, we find that $x_1 = x_2 = x_3$. Thus, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

To find this steady-state, we need to scale the eigenvector of $\mathbf{B}$ corresponding to an eigenvalue of 1 according to the total number of airplanes, which is 900. Scaled, we get:

$$\vec{p}[\infty] = \begin{bmatrix} 300 \\ 300 \\ 300 \end{bmatrix}$$

After infinite timesteps, there is an even distribution of airplanes among the three airports!

8. From Independence to Dependence (12 points)

Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \\ -2 & 4 \end{bmatrix}$ and $\vec{v}_1$ and $\vec{v}_2$ be two vectors in $\mathbb{R}^2$. Prove that the set of vectors $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2\}$ must be linearly dependent.

Solution: There are many possible approaches for this proof. We show one below.

First, let $\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$. Then we have:

$$\begin{aligned} \mathbf{A}\vec{v}_1 &= \begin{bmatrix} 1 & -2 \\ 3 & -6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \\ &= \begin{bmatrix} a_1 - 2b_1 \\ 3a_1 - 6b_1 \\ -2a_1 + 4b_1 \end{bmatrix} \\ &= a_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + b_1 \begin{bmatrix} -2 \\ -6 \\ 3 \end{bmatrix} \\ &= (a_1 - 2b_1) \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \end{aligned} \tag{1}$$

Similarly, we get $\mathbf{A}\vec{v}_2 = (a_2 - 2b_2) \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$. We see that both vectors $\mathbf{A}\vec{v}_1$ and $\mathbf{A}\vec{v}_2$ are scalar multiples of each other. Therefore, the set of vectors $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2\}$ must be linearly dependent.

A common misconception on the exam was that the columns of $\mathbf{A}$ being linearly dependent directly imply $\mathbf{A}\vec{v}_1$ and $\mathbf{A}\vec{v}_2$ are linearly dependent. However, this is not always true.
Take for example, the following matrix and vectors:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2}$$

We see that $\mathbf{B}\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{B}\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent, even though the columns of $\mathbf{B}$ are linearly dependent.

The real conceptual reason for the original problem is that $\mathbf{A}$'s column space only has dimension 1, which is the span of $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$. Any matrix multiplication $\mathbf{A}\vec{v}$ will always be a scalar multiple of $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ since the matrix multiplication represents a linear combination of the column space of $\mathbf{A}$. Thus, any two vectors $\mathbf{A}\vec{v}_1$ and $\mathbf{A}\vec{v}_2$ will always be linearly dependent.

9. Transform that eigenvector! (26 points)

(a) (8 points) Suppose that a matrix $\mathbf{M}$ has eigenvectors $\vec{v}_1, \vec{v}_2$, and corresponding eigenvalues λ_1, λ_2. Consider the matrix $\mathbf{N}$, which performs the transformation performed by $\mathbf{M}$ twice. In other words, for some arbitrary vector, $\vec{u}$, the following holds:

$$(\mathbf{M}^2)\vec{u} = \mathbf{N}\vec{u}$$

What are the eigenvalues and eigenvectors of $\mathbf{N}$, in terms of $\vec{v}_1, \vec{v}_2$, λ_1, and λ_2? Justify your answer.

Solution: First, we recognize that $\mathbf{N}$ is the equivalent of $\mathbf{M}^2$.

For an eigenvector $\vec{v}_i$ of $\mathbf{M}$:

$$\mathbf{M}\vec{v}_i = \lambda_i \vec{v}_i$$
$$\mathbf{M} * (\mathbf{M}\vec{v}_i) = \mathbf{M}(\lambda_i \vec{v}_i)$$
$$\mathbf{M}^2 \vec{v}_i = \lambda_i \mathbf{M}\vec{v}_i$$
$$\mathbf{M}^2 \vec{v}_i = \lambda_i^2 \vec{v}_i$$
$$\mathbf{N} v_i = \lambda_i^2 \vec{v}_i$$

This allows us to deduce the following:

- The eigenvectors of $\mathbf{N}$ are the eigenvectors of $\mathbf{M}$.
- For an eigenvalue λ_i of $\mathbf{M}$, the equivalent eigenvalue for the matrix $\mathbf{N}$ is λ_i^2.

Therefore, $\mathbf{N}$ has eigenvectors $\vec{v}_1$ and $\vec{v}_2$. It has eigenvalues of λ_1^2 and λ_2^2.

(b) (9 points) Consider the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, which takes a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and *scales* its y component by 2, but leaves the x component unchanged. In other words, $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$. What are the eigenvalues and corresponding eigenvectors of $\mathbf{A}$?

Solution: Since only the y component is scaled, we first consider vectors which point purely along the y direction, such as the coordinate vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Based on the information given, we can write this equation:

$$\mathbf{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

From this, we can deduce that $\lambda_1 = 2$ and $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We can also consider the coordinate vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. From the information given, we can write:

$$\mathbf{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We can deduce that $\lambda_2 = 1$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus, $\lambda_1 = 2, \lambda_2 = 1, \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

It's also possible to construct the matrix $\mathbf{A}$ from the information given to solve for the eigenvalues and corresponding eigenvectors. Full credit is awarded for identifying the correct eigenvalue and eigenvector pairs.

(c) (9 points) Now consider the matrix $\mathbf{C} \in \mathbb{R}^{2\times 2}$ that performs the following vector transformations in order:

- counterclockwise rotation by 45 degrees
- scales the y component by 2 and leaves the x component unchanged
- clockwise rotation by 45 degrees

In other words, the matrix transformation can be written as the following matrix multiplication:

$$\mathbf{C}\vec{v} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}\vec{v}$$

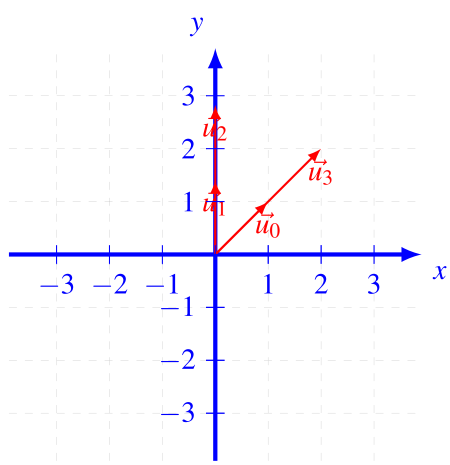
where the matrix $\mathbf{B} \in \mathbb{R}^{2\times 2}$ performs a counterclockwise rotation by 45 degrees, and the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ doubles the y component of the vector.

What are the eigenvalues and corresponding eigenvectors of this matrix $\mathbf{C}$?

Solution: We see that **C** will perform a rotation, scale the y component, then rotate the vector back to it's original position. Similar to the previous part, we want to consider vectors which point purely along the y direction or x direction during the scaling operation. To do this, we need to first start with a vector that is 45 degrees rotated from the coordinate vectors.

First, consider $\vec{u}_0 = \begin{bmatrix}1\\1\end{bmatrix}$. We can step through each operation of the matrix transformation.

$$\vec{u}_0 = \begin{bmatrix}1\\1\end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix}0\\2\sqrt{2}\end{bmatrix} \rightarrow \vec{u}_3 = \begin{bmatrix}2\\2\end{bmatrix}$$



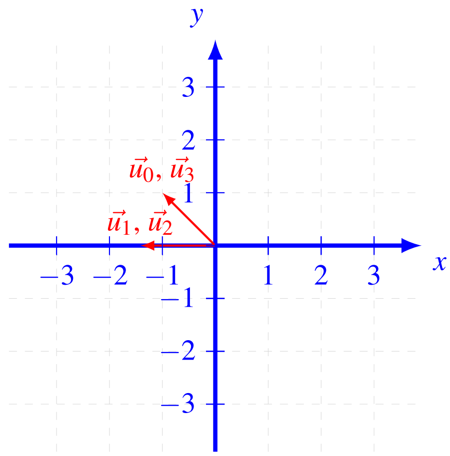
We see that matrix C transforms $\vec{u}_0 = \begin{bmatrix}1\\1\end{bmatrix}$ to $\vec{u}_3 = \begin{bmatrix}2\\2\end{bmatrix}$. Written as a matrix-vector multiplication,

$$\mathbf{C}\begin{bmatrix}1\\1\end{bmatrix} = \lambda_1\begin{bmatrix}2\\2\end{bmatrix}.$$

Thus, $\lambda_1 = 2$ and $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$.

Now, let's look at another similar vector. Let $\vec{u}_0 = \begin{bmatrix}-1\\1\end{bmatrix}$. Again, we can step through each operation of the matrix transformation.

$$\vec{u}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} \to \vec{u}_2 = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} \to \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Notice that $\vec{u}_1$ lies solely on the x-axis. Thus, the initial vector is rotated counterclockwise and rotated back to its original position. In other words, the matrix $\mathbf{C}$ transforms $\vec{u}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to $\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Written as a matrix-vector multiplication,

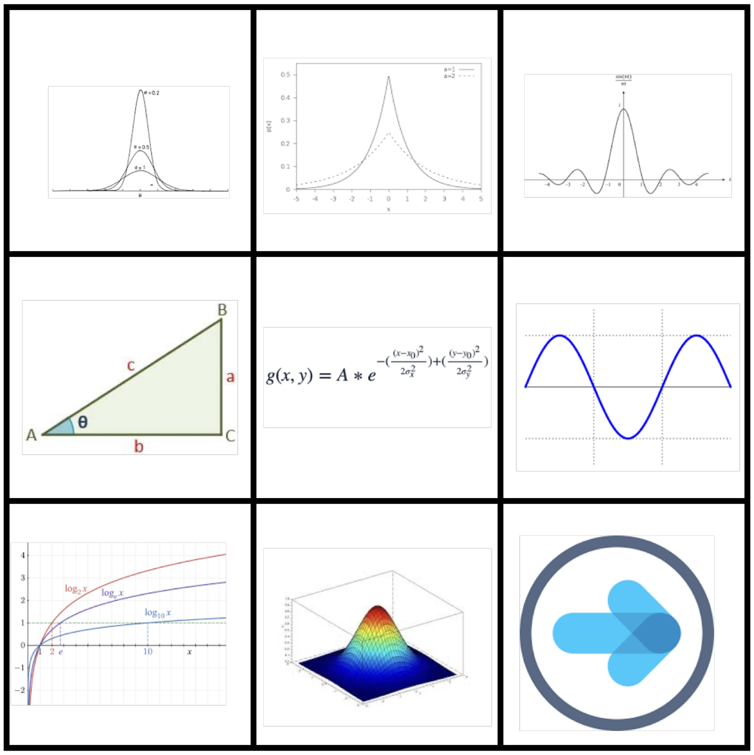
$$\mathbf{C} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus, $\lambda_2 = 1$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Notice that $\vec{v}_1$ and $\vec{v}_2$ can be any vectors in the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. As long as $\vec{v}_1$ is a constant multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2$ is a constant multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and the corresponding eigenvalues are correctly identified as $\lambda_1 = 2$ and $\lambda_2 = 1$, full credit is awarded.

And, similar to the previous part, it's possible to construct the matrix $\mathbf{C}$ from the information given to solve for the eigenvalues and corresponding eigenvectors. Full credit is awarded for identifying the correct eigenvalue and eigenvector pairs.

10. G.E. Game (0 points)



Gaussian Elimination: The game — not the algorithm. Eliminate those Gaussians!

Solution: