EECS 16A Designing Information Devices and Systems I
Fall 2023 Final Exam

Final Solution

PRINT your student ID: ____________________

PRINT your name: ____________________, ____________________
(last name) (first name)

PRINT your discussion section and GSI: ____________________

Name and SID of the person to your left: ____________________

Name and SID of the person to your right: ____________________

Name and SID of the person in front of you: ____________________

Name and SID of the person behind you: ____________________

0. Honor Code (0 Points)
Acknowledge that you have read and agree to the following statement and sign your name below:
As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.
If you do not sign your name, you will get a 0 on the exam.

1. When the exam starts, write your SID at the top of every page. (3 Points)
No extra time will be given for this task.

2. Tell us about something you are proud of this semester. (2 Points)

3. What are you looking forward to over winter break? (2 Points)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

PRINT your name and student ID: ____

4. Strike a Chord (4 points)

Alex built a bot that helps you learn to play the guitar. It listens to you play a melody and compares it to a target melody. Each melody maps to a vector. The target melody you are learning maps to $\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$. You play "melody A" that maps to $\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}^T$ and "melody B" that maps to $\begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}^T$. A smaller angle between two melodies means they are closer. **Does "melody A" or "melody B" have a smaller angle with the target melody?** Justify your response.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

Table 4.1: Helpful cosine values.

Solution:
The formula for the cosine value of an angle between two vectors $\vec{x}$ and $\vec{y}$ is given by

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \, ||\vec{y}||}.$$

Denote target vector as $\vec{t}$, melody A's vector as $\vec{a}$, and melody B's vector as $\vec{b}$. We can find the angles θ_a and θ_b: the angles between the target melody and each of melodies A and B, respectively.

$$\cos(\theta_a) = \frac{\vec{a} \cdot \vec{t}}{||\vec{a}|| \, ||\vec{t}||}$$

$$= \frac{\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T}{\sqrt{(-1)^2 + (1)^2 + (1)^2 + (-1)^2}\sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2}}$$

$$= \frac{-1 - 1 + 1 + 1}{4}$$

$$= 0.$$

According to the table, this corresponds to an angle of 90 degrees for θ_a.

$$\cos(\theta_b) = \frac{\vec{b} \cdot \vec{t}}{||\vec{b}|| \, ||\vec{t}||}$$

$$= \frac{\begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T}{\sqrt{(1)^2 + (-1)^2 + (-1)^2 + (-1)^2}\sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2}}$$

$$= \frac{1 + 1 - 1 + 1}{4}$$

$$= \frac{1}{2}.$$

According to the table, this corresponds to an angle of 60 degrees for θ_b.

Since $\theta_b < \theta_a$, melody B is closer to the target melody.

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5. Landing Gear (13 points)

Youbin, a forgetful space pilot, often forgets to deploy his landing gear on his spaceship. Using his knowledge from the capacitive touchscreen lab, he wants to design a circuit that will sense when the spaceship is close to the surface and automatically deploy the landing gear.

(a) (5 points) Youbin installs two electrodes E_1 and E_2 on the bottom of his spaceship as shown in Figure 5.1.

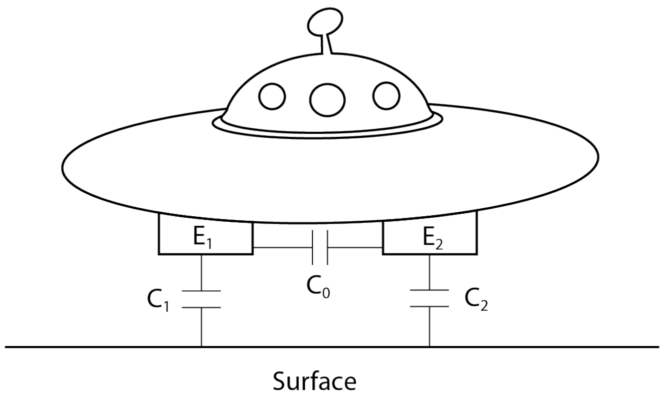
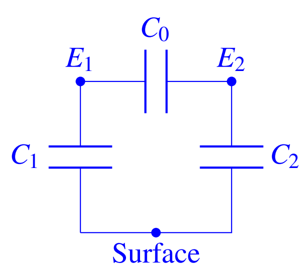


Figure 5.1: Capacitance diagram when landing

The two electrodes form a capacitor with capacitance C_0. When the spaceship nears the surface, the electrodes also form a capacitor C_1 and C_2 with the surface. The surface can be assumed to be conductive. **Draw a circuit diagram that represents the system when the spaceship is near the surface. Explicilty label the capacitors C_0, C_1, C_2 and the nodes E_1, E_2 and Surface. What is the equivalent capacitance $C_{\mathbf{eq}}$ between E_1 and E_2 when the spaceship is near the surface?** You may use the parallel operator in your answer.

Solution: Since the surface is conducting, the bottom plate of C_1 is connected to the bottom plate of C_2 and we can draw the following circuit diagram that represents the capacitance when the spaceship is near the surface:



With respect to nodes E_1 and E_2, we can use our equivalent capacitance equations to note that

$$C_{eq} = C_0 + (C_1 \parallel C_2).$$

(b) (4 points) In order to detect the change in capacitance, Youbin connects a time-varying current source $I_s(t)$ to the electrodes with effective capicitance C_{eq}, as shown in Figure 5.2.

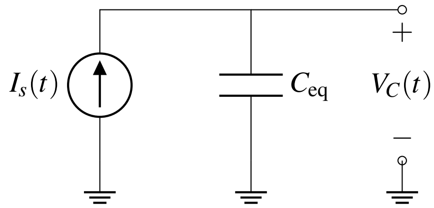


Figure 5.2

He knows that when landing, $C_{\text{eq}} = 1\,\mu\text{F}$ and $I_s(t)$ outputs a square wave shown in Figure 5.3.

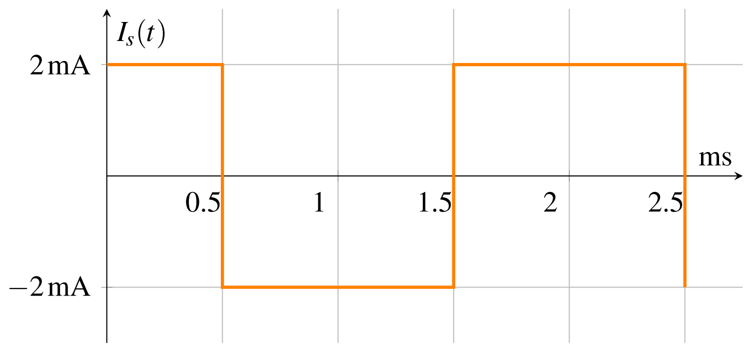


Figure 5.3

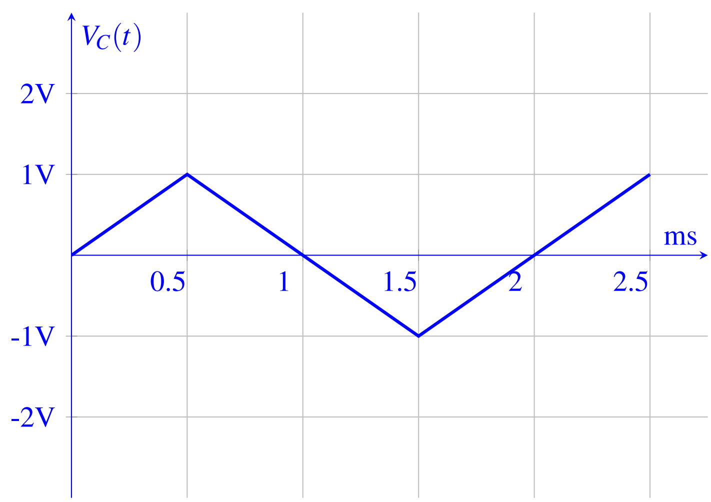
Assuming $V_C(0) = 0\,\text{V}$, **plot** $V_C(t)$ **from** $t = 0\,\text{ms}$ **to** $t = 2.5\,\text{ms}$ **in the space provided below. Clearly label the minimum and maximum values.**

Solution:

For a capacitor, we know that $Q = C_{\text{eq}} V_C$. Taking the derivative of both sides, we see that $I_s = \frac{dQ}{dt} = C_{\text{eq}} \frac{dV_C}{dt}$. In our case, we can solve to find

$$\frac{dV_C}{dt} = \frac{I_s}{C_{\text{eq}}} = \frac{\pm 2\,\text{mA}}{1\,\mu\text{F}} = \pm 2\,\text{V/s}\,,$$

which means that V_C will change linearly in a triangle wave. Knowing that $V_C(0) = 0\,\text{V}$, we can draw



(c) (4 points) Youbin finds that the difference in peak voltages of V_C is small when the capacitance changes. He decides to amplify V_C by a gain of 5 in order to better distinguish the peak voltages. He designs the circuit shown in Figure 5.4. You may assume the op-amp is ideal. **Choose values for resistors R_1 and R_2 such that $V_{out} = 5V_C$.** Show your work.

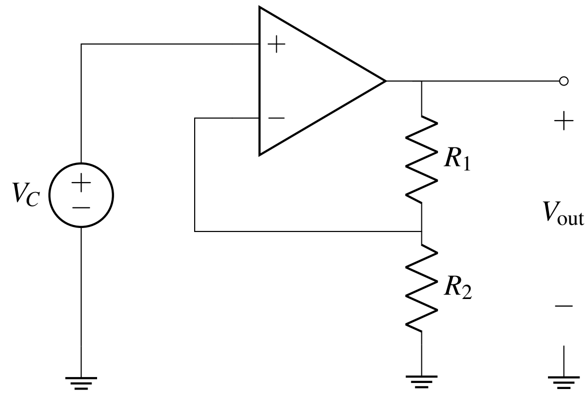


Figure 5.4: Amplifier circuit

Solution: We notice that the circuit drawn is a non-inverting op-amp. As such, we can note that

$$\frac{V_{out}}{V_C} = 1 + \frac{R_1}{R_2}.$$

For this problem, we want

$$5 = 1 + \frac{R_1}{R_2}$$
$$\frac{R_1}{R_2} = 4.$$

For example, we can pick $R_1 = 4\,\mathrm{k}\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, although any combination of R_1 and R_2 that satisfies the ratio above is a valid solution.

PRINT your name and student ID: ____

6. Let's Go, Mooncow! (28 points)

UC Berkeley, in preparation for their new Space Exploration Research Center at NASA Ames, has tasked you with understanding the space travels of a newly discovered creature named "Mooncow".

(Despite his name, Mooncow bears a surprisingly strong resemblance to what we call "Monkeys" on Earth).

A scientist's depiction of "Mooncow"

For this problem, assume the galaxy is two-dimensional, and the sun represents the origin.

(a) (2 points) Mooncow is moving in a 2D galaxy and has access to three boosters. Each booster moves him in a specific direction: $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$. **He must choose the fewest number of boosters to reach any point in the galaxy. Which boosters should he choose?**

Note: There may be multiple correct answers.

☐ $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

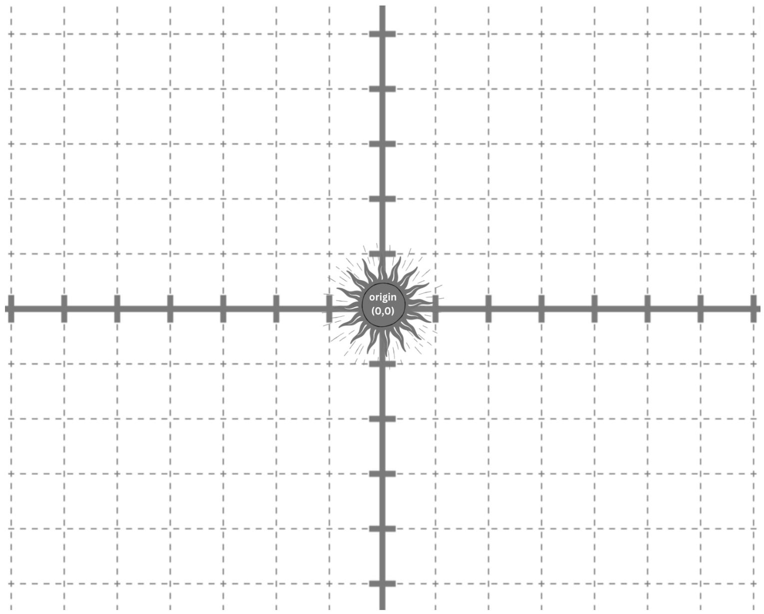
☐ $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$

☐ $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

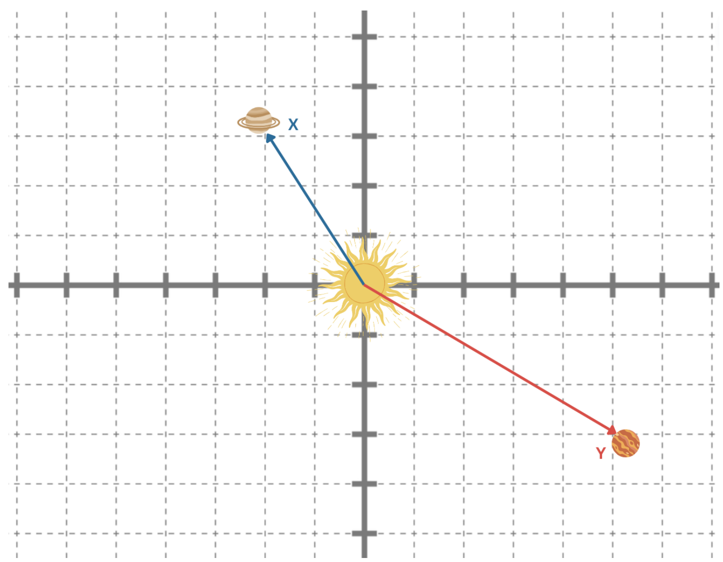
Solution:

Mooncow needs to choose boosters corresponding to 2 linearly independent vectors to span all of $\mathbb{R}^2$. Using the definition of linear independence, the possible solutions are $\{\begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\}$ or $\{\begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\}$.

(b) (3 points) Mooncow wants to plot the locations of two planets. Using the provided graph, **plot the position vectors of the planets he sees: Planet X: $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, Planet Y: $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$. Label the planets.**



Solution:



(c) (4 points) Mooncow sees a solar eclipse taking place on Planet B due to the position of Planet A. He is at $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, Planet A is at $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Planet B is at $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$. Mooncow wants to travel to the eclipsed region (i.e. the line segment joining the two planets) as shown in Figure 6.1. Mooncow takes the shortest path to reach this line segment.

Compute the coordinates of where Mooncow will arrive on the path of the eclipse, and state how far he will be from Planet A when he arrives. Your solution must be justified by calculations, but you may use the graph to help you.

Solution:

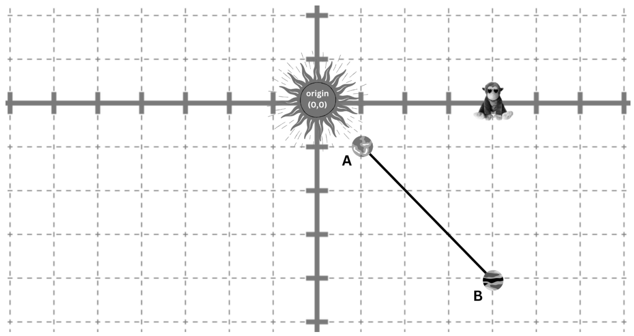
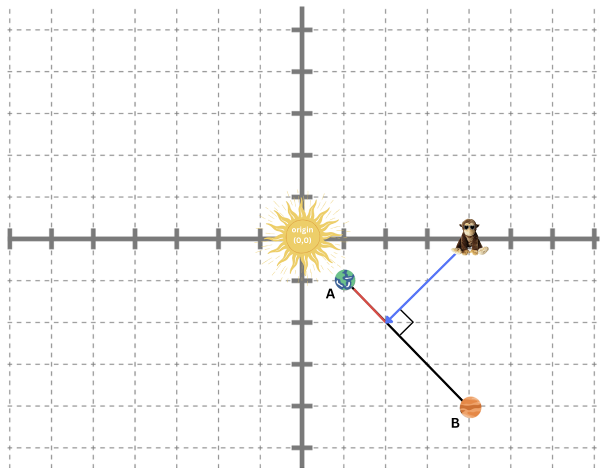


Figure 6.1



Mooncow arrives at $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$, which is $\sqrt{2}$ units from Planet A.

Justification:

i. Calculate $\vec{u} = \vec{b} - \vec{a}$, corresponding to the line segment between Planet A and Planet B:

$$\vec{u} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

ii. Denote Mooncow's position vector as $\vec{v}$ and calculate the dot product $\vec{v} \cdot \vec{u}$:

$$\vec{v} \cdot \vec{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \end{bmatrix} = (4 \cdot 3) + (0 \cdot -3) = 12$$

iii. Calculate the squared norm of $\vec{u}$, $\|\vec{u}\|^2$:

$$\|\vec{u}\|^2 = \left\| \begin{bmatrix} 3 \\ -3 \end{bmatrix} \right\|^2 = 3^2 + (-3)^2 = 9 + 9 = 18$$

iv. Use the projection formula to find $\text{proj}_{\vec{u}}(\vec{v})$:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \cdot \vec{u} = \frac{12}{18} \cdot \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \frac{2}{3} \cdot \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Alternative solution - geometric approach:
The slope of the line segment between the two planets is -1, so the shortest distance between the line segment and Mooncow is an orthogonal line that passes through Mooncow's location.

This means the shortest path has a slope of 1 and passes through $\begin{bmatrix}4\\0\end{bmatrix}$.

Using point-slope form, the equation for this line is $(y-0) = 1(x-4)$, which simplifies to $y = x-4$
And the equation for the line segment between the two planets is $(y-1) = -1(x+1)$, which simplifies to $y = -x$

The intersection of these two lines is at $\begin{bmatrix}2\\-2\end{bmatrix}$, so this is where Mooncow will arrive.

Lastly, the distance from Planet A is given by finding the magnitude of the difference between Planet A's position and Mooncow's position: $\begin{bmatrix}2\\-2\end{bmatrix}$ - $\begin{bmatrix}1\\-1\end{bmatrix}$ = $\begin{bmatrix}1\\-1\end{bmatrix}$

$$d = \sqrt{(2-1)^2+(-2-(-1))^2} = \sqrt{1^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$$

(d) (3 points) Mooncow's position vector is at $\begin{bmatrix}2\\4\end{bmatrix}$. He is orbiting the sun in a counterclockwise direction. His velocity vector is in the direction of his motion and is orthogonal to his position vector. **Calculate Mooncow's velocity vector. His velocity vector should be unit length.** Show your work.

Solution:

Mooncow's position vector $\vec{x}$ is given by $\begin{bmatrix}2\\4\end{bmatrix}$.

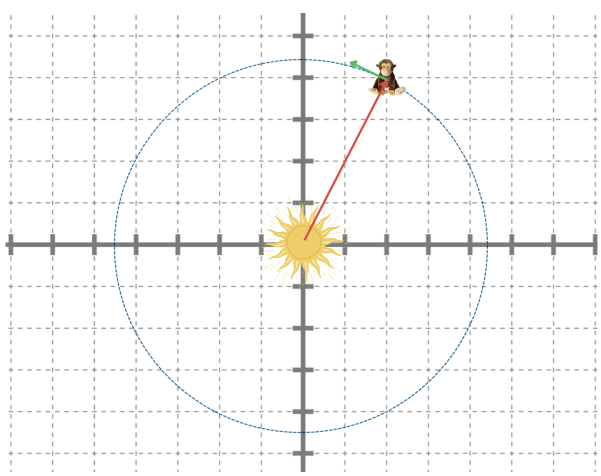
His velocity vector, denoted as $\vec{v}$, is orthogonal to his position vector, so $\vec{x}\cdot\vec{v} = 0$.

Let $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$. Thus, the dot product $\vec{x}\cdot\vec{v} = 0$ gives us the equation $2v_1 + 4v_2 = 0$.

Solving this equation leads to $\vec{v} = \begin{bmatrix}-2\\1\end{bmatrix}$ (or some scalar multiple of this vector).

Magnitude of this vector is $\sqrt{(-2)^2+(1)^2} = \sqrt{5}$

Therefore, the corresponding unit vector is: $\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$.



(e) (4 points) Kanav finds Mooncow is located at $\begin{bmatrix}4\\6\end{bmatrix}$. He recalibrates his measurement device and finds these measurements need to be rotated **clockwise** by 60 degrees. **Find Mooncow's real location.** Show your work. Recall that $\sin(60°) = \frac{\sqrt{3}}{2}$, $\sin(-60°) = \frac{-\sqrt{3}}{2}$, and $\cos(60°) = \cos(-60°) = \frac{1}{2}$.

Solution:

Rotation matrix is given by:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

We are applying a clockwise rotation, so we will use the rotation matrix with $\theta = -60$ degrees. This gives us the rotation matrix R:

$$R = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

.

Apply this rotation matrix to the initial measured point: $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2+3\sqrt{3} \\ 3-2\sqrt{3} \end{bmatrix}$

Thus, Mooncow's real location is at $\begin{bmatrix} 2+3\sqrt{3} \\ 3-2\sqrt{3} \end{bmatrix}$.

(f) (4 points) We have lost track of Mooncow and are searching the galaxy for him! Anish is located on Planet X at $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$, and he detects Mooncow is 4 units away. Sabriya is located on Planet Y at $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$, and she detects Mooncow is 5 units away. They know Mooncow always stays at least 2 units away from the sun. **What coordinates is Mooncow at?** Show your work. Your solution must be justified by calculations, but you may use the graph to help you.

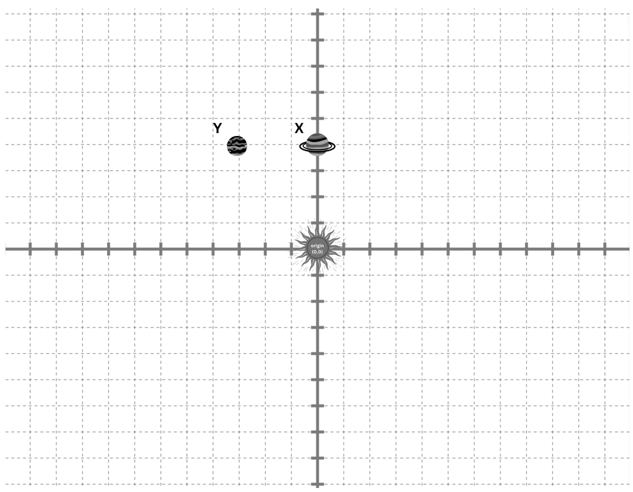


Figure 6.2

Solution:

Planet X is centered at $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and Mooncow is detected 4 units away, so the equation corresponding to his possible locations is $(x)^2 + (y-4)^2 = 16$.

Planet Y is centered at $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and Mooncow is detected 5 units away, so the corresponding equation for his possible locations is $(x-(-3))^2 + (y-4)^2 = 25$.

Solving the system of equations:

$$(x)^2 + (y-4)^2 = 16$$
$$(x-(-3))^2 + (y-4)^2 = 25$$

Expand and simplify each equation:

$$x^2 + y^2 - 8y + 16 = 16$$
$$x^2 + 6x + 9 + y^2 - 8y + 16 = 25$$

:

$$x^2 + y^2 - 8y = 0$$
$$x^2 + 6x + y^2 - 8y = 0$$

Subtracting first equation from the second:

$$6x = 0$$
$$x = 0$$

Substituting this value back into $x^2 + y^2 - 8y = 0$:

$$y^2 - 8y = 0$$
$$y = 0, 8$$

So the possible locations for Mooncow are given by the intersection points: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$. Since we know Mooncow must be at least 2 units away from the sun, he must be located at $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$.

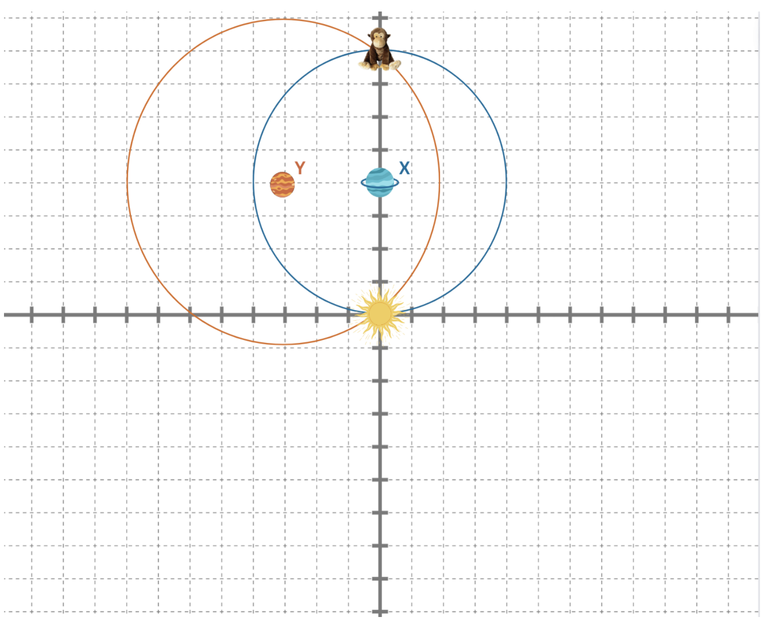


Table 6.1

x	y
-3	-8
0	10
5	0
0	-10
-5	0
4	6

(g) (4 points) Sayan has been tracking Mooncow and has the following measurements for Mooncow's positions:

Kepler's laws dictate that Mooncow's spaceship follows an elliptical orbit. Recall an ellipse follows the formula $\alpha x^2+\beta xy+\gamma y^2+\delta x+\varepsilon y=1$. **What are the unknowns Sayan must identify to find the equation for the ellipse? Using the data points in Table 6.1, formulate the least squares equation in matrix-vector form that would be used to solve for the equation of the ellipse.**

Solution:

The unknowns are $\alpha, \beta, \gamma, \delta$, and ε.

We can formulate the least squares problem as follows:

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 \\ x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} (-3)^2 & (-3)(-8) & (-8)^2 & -3 & -8 \\ (0)^2 & (0)(10) & (10)^2 & 0 & 10 \\ (5)^2 & (5)(0) & (0)^2 & 5 & 0 \\ (0)^2 & (0)(-10) & (-10)^2 & 0 & -10 \\ (-5)^2 & (-5)(0) & (0)^2 & -5 & 0 \\ (4)^2 & (4)(6) & (6)^2 & 4 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 24 & 64 & -3 & -8 \\ 0 & 0 & 100 & 0 & 10 \\ 25 & 0 & 0 & 5 & 0 \\ 0 & 0 & 100 & 0 & -10 \\ 25 & 0 & 0 & -5 & 0 \\ 16 & 24 & 36 & 4 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(h) (4 points) To bring Mooncow back home, Anish needs to know Mooncow's mass. He cannot measure Mooncow's mass directly; instead, he measures the gravitational force on Mooncow F and Mooncow's acceleration a and uses the equation $F = ma$ to solve for mass. The measurements of Mooncow's acceleration and force are as follows:

Table 6.2

$a(\frac{m}{s^2})$	$F(\frac{kg \cdot m}{s^2})$
-2	-20
-1	-15
0	-3
1	10
2	20

We use the equation $F \approx ma$ to relate these variables. **Set up a least squares problem to estimate m. Compute the least squares solution of m.** Show your work.

Solution:

We formulate the least squares problem as follows:

$$A\vec{x} = \vec{b}$$

In this case, A maps to $\vec{a}$, $\vec{x}$ maps to $\vec{m}$, and $\vec{b}$ maps to $\vec{F}$, leading to the following setup:

$$\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \vec{m} = \begin{bmatrix} -20 \\ -15 \\ -3 \\ 10 \\ 20 \end{bmatrix}$$

We then can compute the value of $\hat{m}$ using the formula $\hat{x} = (A^T A)^{-1} A^T \vec{b}$:

$$\hat{m} = (\begin{bmatrix} -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix})^{-1} ([-2 \quad -1 \quad 0 \quad 1 \quad 2)] \begin{bmatrix} -20 \\ -15 \\ -3 \\ 10 \\ 20 \end{bmatrix}$$

$$\hat{m} = (10)^{-1} \cdot (40 + 15 + 0 + 10 + 40) = 10.5\ kg$$

PRINT your name and student ID: ____

7. Caterwauls! (18 points)

(a) (4 points) Thomas' cat Luna frequently wanders off. In order to keep track of her, Thomas is building a tracking system. He installs a tracking collar that transmits a distinct signal $\vec{l}$ shown in Figure 7.1.

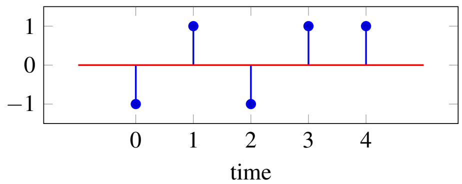


Figure 7.1: Luna's signal $\vec{l}$

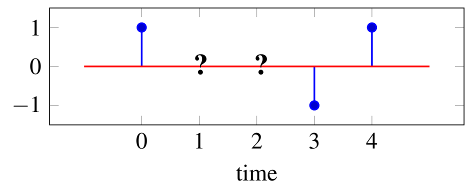


Figure 7.2: Recorded signal $\vec{r}$

In order to test his system he records the signal $\vec{r}$, as well as the cross-correlation $\text{corr}_{\vec{r}}(\vec{l})$. Unfortunately, he realizes that $\vec{r}$ has been corrupted in some places, as shown in Figure 7.2. The cross-correlation $\text{corr}_{\vec{r}}(\vec{l})$ is given in Figure 7.3.

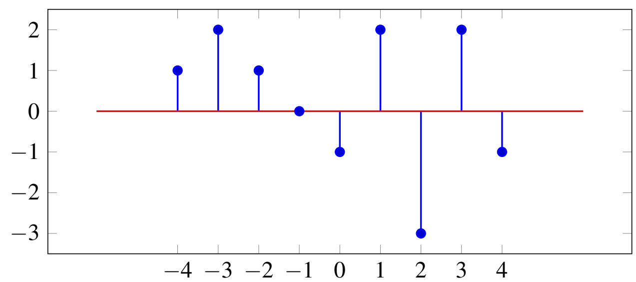


Figure 7.3: Cross-correlation $\text{corr}_{\vec{r}}(\vec{l})$

Recover the missing entries $\vec{r}[1]$ **and** $\vec{r}[2]$. Show your work.

Solution:

Let us represent the recorded signal $\vec{r}$ as $[1, a, b, -1, 1]$ with a and b being our unknown values. We can compute the cross-correlation $\text{corr}_{\vec{r}}(\vec{l})[k]$ at $k = 1$ and $k = 2$ to get

$$\text{corr}_{\vec{r}}(\vec{l})[1] = -a + b + 1 + 1 = -a + b + 2$$
$$\text{corr}_{\vec{r}}(\vec{l})[2] = -b - 1 - 1 = -b - 2$$

Using values from cross-correlation plot at shift values of 1 and 2, we get that

$$\text{corr}_{\vec{r}}(\vec{l})[1] = 2 = -a + b + 2$$
$$\text{corr}_{\vec{r}}(\vec{l})[2] = -3 = -b - 2.$$

The system of equations has solution $a = 1$ and $b = 1$. Therefore we get that $\vec{r}[1] = 1$ and $\vec{r}[2] = 1$.

(b) (3 points) Luna has wandered off! To locate her, Thomas records the signal $\vec{s}$ transmitted by Luna's collar and computes $\text{corr}_{\vec{s}}(\vec{l})$ shown in Figure 7.4.

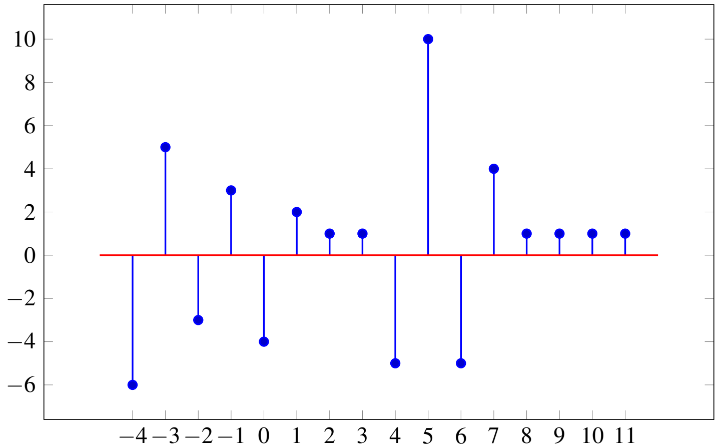


Figure 7.4: Cross-correlation $\text{corr}_{\vec{s}}(\vec{l})$

Assume that the x-axis ticks correspond to a shift of 1×10^{-6} s and the transmissions travel at 3×10^{8} m/s. **Compute the distance between Thomas and Luna**. Show your work.

Solution:

The peak of the cross-correlation occurs at $k = 5$, so we know the time delay between the Luna and Thomas is 5×10^{-6} s. Therefore we know that Luna must be $5 \times 10^{-6}\,\text{s} \cdot 3 \times 10^{8}\,\text{m/s} = 1.5\,\text{km}$ away from Thomas.

(c) (4 points) To prevent Luna from wandering too far, Thomas wants to design a circuit that plays a recall sound through a speaker. The volume of the speaker should increase proportionally to Luna's distance from him. Thomas already has a converter circuit that converts Luna's distance to a voltage V_{dist}. The converter circuit and speaker can be represented by the Thevenin equivalents shown in Figures 7.5 and 7.6 respectively.

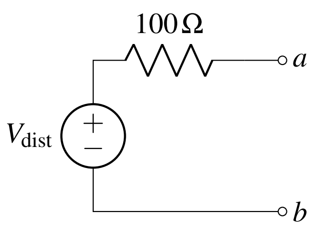


Figure 7.5: Thevenin equivalent of converter

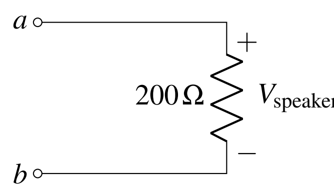


Figure 7.6: Thevenin equivalent of speaker

Thomas first connects the two circuits together directly, shown in Figure 7.7.

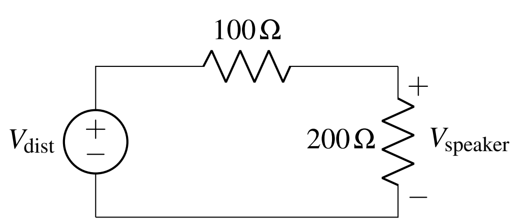


Figure 7.7: Direct connection

Given that $0\,\mathrm{V} \leq V_{\text{dist}} \leq 6\,\mathrm{V}$, **what is the maximum power dissipated by the speaker?** Show your work.

Solution:

Using the voltage divider formula, we notice that

$$V_{\text{speaker}} = \frac{200\,\Omega}{200\,\Omega + 100\,\Omega} V_{\text{dist}} = \frac{2}{3} V_{\text{dist}}.$$

We can compute the power dissipated by a resistor as

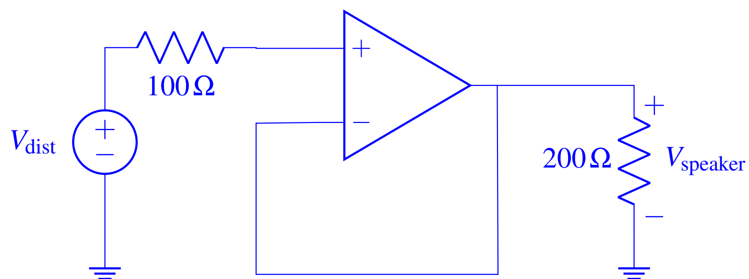
$$P_{\text{speaker}} = I V_{\text{speaker}} = \frac{V_{speaker}^2}{200\,\Omega} = \frac{4}{9} \frac{V_{\text{dist}}^2}{200\,\Omega}.$$

Given the possible ranges of V_{dist}, the maximum power will occur when $V_{\text{dist}} = 6\,\mathrm{V}$. In this case, the maximum power dissipated by the speaker is

$$P_{\text{speaker}} = \frac{4}{9} \frac{36\,\mathrm{V}^2}{200\,\Omega} = \frac{2}{25}\,\mathrm{W}.$$

(d) (4 points) Thomas realizes that the speaker volume is too low when directly connected to the converter. He instead wants to connect the circuits such that $V_{\text{speaker}} = V_{\text{dist}}$. He only has access to a single ideal op-amp and no other components. **Complete the circuit below by connecting the elements given. No element terminal should be left unconnected.**

Solution: We can use the op-amp buffer in order to make sure that the voltage from the input is preserved and not affected by the resistors. The buffer is connected as follows:



(e) (3 points) Thomas needs to build a resistor out of resistive cubes which have a length, width, and height of $5 \times 10^{-3}\,\mathrm{m}$ and a resistivity of $8 \times 10^{-3}\,\Omega\mathrm{m}$. He plans to attach the cubes in a line into one long resistor. **How many cubes does he need to make a** $40\,\Omega$ **resistor?** Justify your answer.

Solution: Using the formula $R = \frac{\rho L}{A}$ to compute the resistance, we find that the resistance of one cube is:

$$R_{\text{cube}} = 8 \times 10^{-3}\,\Omega\mathrm{m} \frac{5 \times 10^{-3}\,\mathrm{m}}{5 \times 10^{-3}\,\mathrm{m} \cdot 5 \times 10^{-3}\,\mathrm{m}} = \frac{8}{5}\,\Omega.$$

If we line up n cubes, they form a chain of resistors in series with total resistance $R = nR_{\text{cube}}$. Therefore we need

$$n = \frac{R}{R_{\text{cube}}} = \frac{40\,\Omega}{\frac{8}{5}\,\Omega} = 25$$

cubes.

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8. Aficiona-dough (25 points)

Jiarui owns two pizza shops: Slice and Cheddarboard. He models the movement of his customers each week. Each timestep represents a week.

(a) (4 points) Each week 40% of Slice's customers move to Cheddarboard to buy pizza, while the remaining customers stay at Slice. 25% of Cheddarboard customers move to Slice, while the remainder stay at Cheddarboard. **Draw a state transition diagram modeling the flow of customers between Jiarui's restaurants.**

Solution:

The state transition diagram is shown in Figure 8.1.

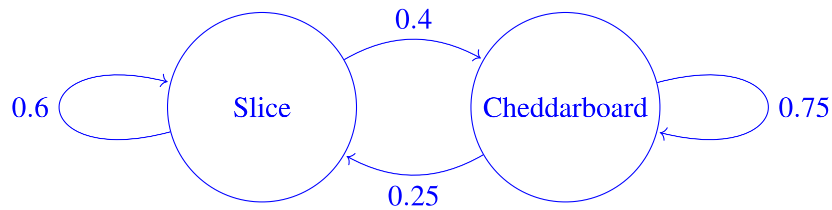

Figure 8.1: State transition diagram of the system.

(b) (4 points) Jiarui observes that the system follows a new state transition diagram (due to a change in his menu), which is given in Figure 8.2.

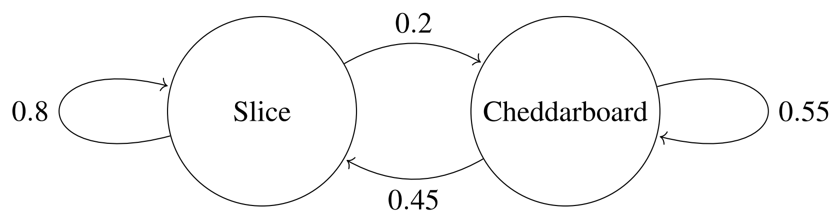

Figure 8.2: New state transition diagram of the system.

Write the state transition matrix $\mathbf{P}$ corresponding to the Figure 8.2, such that $\vec{c}[t+1] = \mathbf{P} \cdot \vec{c}[t]$ where $\vec{c}[t] = \begin{bmatrix} Slice[t] \\ Cheddarboard[t] \end{bmatrix}$.

What is the nullspace of P? Justify your answer.

Hint: You need not mathematically compute the nullspace.

Solution:

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.45 \\ 0.2 & 0.55 \end{bmatrix}.$$

$\mathbf{P}$ has linearly independent columns. Hence it has a trivial nullspace given by span $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(c) (6 points) Jiarui opens a third pizza shop: Asparagus. Initially, he has a total of 120 customers. The state transition matrix of the system describing the flow of customers between the three restaurants is:

$$\mathbf{Q} = \begin{bmatrix} \frac{2}{3} & \frac{2}{5} & \frac{1}{2} \\ 0 & \frac{3}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} \end{bmatrix}$$

such that, $\vec{c}[t+1] = \mathbf{Q} \cdot \vec{c}[t]$ where $\vec{c}[t] = \begin{bmatrix} Slice[t] \\ Cheddarboard[t] \\ Asparagus[t] \end{bmatrix}$.

Find the number of customers in each shop at steady state. Show your work.

Solution:

At steady state, $\vec{c^*} = \mathbf{Q} \cdot \vec{c^*}$, which indicates that the steady state vector lies in the eigenspace corresponding to the eigenvalue of 1.

For computing the eigenvector for $\lambda = 1$:

$$\left[\begin{array}{ccc|c} \frac{2}{3}-1 & \frac{2}{5} & \frac{1}{2} & 0 \\ 0 & \frac{3}{5}-1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2}-1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -\frac{1}{3} & \frac{2}{5} & \frac{1}{2} & 0 \\ 0 & -\frac{2}{5} & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{2} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -\frac{6}{5} & -\frac{3}{2} & 0 \\ 0 & -\frac{2}{5} & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{2} & 0 \end{array}\right] \qquad R_1 \leftarrow R_1 \times (-3)$$

$$\to \left[\begin{array}{ccc|c} 1 & -\frac{6}{5} & -\frac{3}{2} & 0 \\ 0 & -\frac{2}{5} & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 \end{array}\right] \qquad R_3 \leftarrow R_3 - \left(\frac{R_1}{3}\right)$$

$$\to \left[\begin{array}{ccc|c} 1 & -\frac{6}{5} & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 \end{array}\right] \qquad R_2 \leftarrow R_2 \times \left(-\frac{5}{2}\right)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{6}{5} & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 \leftarrow R_3 - \left(\frac{2R_2}{5}\right)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_1 \leftarrow R_1 - \left(-\frac{6R_2}{5}\right)$$

Since the third column has no pivot, let $v_3 = t \in \mathbb{R}$ be the free variable. Now we can solve for v_1, v_2 as follows:

$$\begin{aligned} v_1 - \frac{3}{2}v_3 &= 0 \\ v_2 &= 0 \end{aligned} \implies \begin{aligned} v_1 &= \frac{3}{2}t \\ v_2 &= 0 \end{aligned}$$

Writing the above in vector forms gives

$$\vec{v} = \begin{bmatrix} \frac{3}{2}t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} t$$

which indicates that the steady state eigenvector is $\begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$, and the eigenspace is span $\left\{ \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$.

Let $\vec{c^*} = \alpha \cdot \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$, where $\alpha \in \mathbb{R}$.

It can observed that this is a conservative system. So, the total number of customers at each time step is fixed, which is 120 from the initial step.
Solving for $\alpha \cdot (\frac{3}{2} + 0 + 1) = 120$, we get $\alpha = 48$. Hence,

$$\vec{c^*} = \begin{bmatrix} 72 \\ 0 \\ 48 \end{bmatrix}$$

(d) (6 points) Jiarui again observes a change in his system, and finds the new state transition matrix is

$$\mathbf{R} = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{4}{3} & \frac{1}{3} & \frac{4}{3} \end{bmatrix}.$$

Show that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of this matrix. What is the corresponding eigenvalue?

Assuming that the initial state is $\begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$, **how many customers are there in each shop after** 10 **timesteps?** You do not need to reduce your answer. Show your work.

Solution:

If the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{R}$, then the matrix-vector multiplication $\mathbf{R}\vec{v}$ should yield a scalar multiple of $\vec{v}$.

$$\mathbf{R}\vec{v} = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{4}{3} & \frac{1}{3} & \frac{4}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot \left(\frac{5}{3} + \frac{2}{3} + \frac{2}{3}\right) \\ 1 \cdot \left(\frac{4}{3} + \frac{4}{3} + \frac{1}{3}\right) \\ 1 \cdot \left(\frac{4}{3} + \frac{1}{3} + \frac{4}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 3\vec{v}.$$

Hence, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{R}$, with the eigenvalue 3.

The initial state is $100\vec{v}$, which lies in the eigenspace corresponding to eigenvalue 3. So, after each timestep, the number of customers will increase by a factor of 3. Therefore, after 10 timesteps, the number of customers in each shop will be $3^{10} \cdot 100$.

(e) (5 points) Let a state transition matrix $\mathbf{S}$ have eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = \frac{1}{2}$ corresponding to eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively. The initial state is given by

$$\vec{c}[0] = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Let $\vec{c}[t]$ represent the state after t timesteps.
Write $\vec{c}[t]$ in terms of α_i, λ_i and $\vec{v}_i$, where $i = 1, 2, 3$.
Under what conditions on α_1, α_2 and α_3 is $\lim\limits_{t\to\infty} \vec{c}[t]$ finite? Justify your answer.

Solution:

$$\begin{aligned}
\vec{c}[t] &= \mathbf{S}^t \vec{c}[0] \\
&= \mathbf{S}^t \cdot (\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3) \\
&= \alpha_1 \mathbf{S}^t \vec{v}_1 + \alpha_2 \mathbf{S}^t \vec{v}_2 + \alpha_3 \mathbf{S}^t \vec{v}_3 \\
&= \alpha_1 \lambda_1^t \vec{v}_1 + \alpha_2 \lambda_2^t \vec{v}_2 + \alpha_3 \lambda_3^t \vec{v}_3.
\end{aligned}$$

As $t \to \infty$, $\alpha_1 \lambda_1^t \vec{v}_1$ and $\alpha_3 \lambda_3^t \vec{v}_3$ will both be finite since $\alpha_1, \alpha_3 \in \mathbb{R}$, $\lambda_1^t = 1$ and $\lambda_3^t \to 0$. Since $\lambda_2^t \to \infty$, for $\alpha_2 \lambda_2^t \vec{v}_2$ to be finite, α_2 must be 0.
Therefore, for $\lim\limits_{t\to\infty} \vec{c}[t]$ to be finite, $\alpha_2 = 0$.

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9. Proofs (15 points)

(a) (7 points) Consider matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$. Assume that $\mathbf{A}$ is invertible and $\mathbf{B}$ has a nontrivial nullspace. **Prove that BA has a nontrivial nullspace.**

Solution:

Known:

$\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$.

$\mathbf{A}$ is invertible.

$\mathbf{B}$ has a nontrivial nullspace, which means there exists a nonzero vector $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{B}\vec{x} = \vec{0}$.

To show:

$\mathbf{BA}$ has a nontrivial nullspace, which means there exists a nonzero vector $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{BA}\vec{y} = \vec{0}$.

Proof:

Since $\mathbf{A}$ is invertible, we know that both $\mathbf{A}$ and $\mathbf{A}^{-1}$ have a trivial nullspace. We know that $\vec{x}$ is a nonzero vector in the null space of $\mathbf{B}$. Let $\vec{y} = \mathbf{A}^{-1}\vec{x}$. Since $\mathbf{A}^{-1}$ has a trivial nullspace and $\vec{x} \neq \vec{0}$, we know that $\vec{y} \neq \vec{0}$. Next, using the associative property of matrix multiplication, note that

$$\begin{aligned}\mathbf{BA}\vec{y} &= \mathbf{BA}(\mathbf{A}^{-1}\vec{x})\\ &= \mathbf{B}(\mathbf{A}^{-1}\mathbf{A})\vec{x}\\ &= \mathbf{B}\vec{x}\\ &= \vec{0}.\end{aligned}$$

Thus, $\vec{y}$ is a nonzero vector in the nullspace of $\mathbf{BA}$ which proves that $\mathbf{BA}$ has a nontrivial nullspace.

(b) (8 points) Let $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_k$ be k matrices in $\mathbb{R}^{n\times n}$. Assume all $\mathbf{A}_i$ have $\vec{v} \in \mathbb{R}^n$ as an eigenvector, with corresponding eigenvalue λ_i for $i = 1, 2, \cdots, k$. Assume that $\sum_{i=1}^{k} \lambda_i \neq 1$, and the matrix $(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)$ is invertible, where $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. **Prove that $\vec{v}$ is an eigenvector of $(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}$. What is the corresponding eigenvalue?** Show your work.

Solution: **Known:**

$\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_k$ are k matrices in $\mathbb{R}^{n\times n}$.

All $\mathbf{A}_i$ have $\vec{v} \in \mathbb{R}^n$ as an eigenvector, with corresponding eigenvalue λ_i for $i = 1, 2, \cdots, k$, i.e., $\mathbf{A}_i\vec{v} = \lambda_i\vec{v}$.

$\sum_{i=1}^{k} \lambda_i \neq 1$.

$(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}$ exists.

To show:

$\vec{v}$ is an eigenvector of $(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}$, i.e., $(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}\vec{v} = \lambda\vec{v}$, where λ is the corresponding eigenvalue that needs to be computed.

Proof:

Adding the equations $\mathbf{A}_i\vec{v} = \lambda_i\vec{v}$ for $i = 1, 2, \cdots, k$, we get $\sum_{i=1}^{k} \mathbf{A}_i\vec{v} = \sum_{i=1}^{k} \lambda_i\vec{v}$.
Using $\mathbf{I}\vec{v} = 1 \cdot \vec{v}$,

$$\mathbf{I}\vec{v} - \sum_{i=1}^{k} \mathbf{A}_i\vec{v} = 1 \cdot \vec{v} - \sum_{i=1}^{k} \lambda_i\vec{v}$$
$$(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)\vec{v} = (1 - \sum_{i=1}^{k} \lambda_i)\vec{v}.$$

Multiplying both sides by $(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}$,

$$(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)\vec{v} = (\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}(1 - \sum_{i=1}^{k} \lambda_i)\vec{v}$$
$$\implies \mathbf{I}\vec{v} = (1 - \sum_{i=1}^{k} \lambda_i)(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}\vec{v}$$
$$\implies \frac{1}{1 - \sum_{i=1}^{k} \lambda_i}\vec{v} = (\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}\vec{v}.$$

Therefore, $\vec{v}$ is an eigenvector of $(\mathbf{I} - \sum_{i=1}^{k} \mathbf{A}_i)^{-1}$ with corresponding eigenvalue $\frac{1}{1 - \sum_{i=1}^{k} \lambda_i}$.

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10. Orthonormal Least Squares (13 points)

(a) (5 points) Suppose we are given the matrix

$$\mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix},$$

where $||\vec{a}_1|| = ||\vec{a}_2|| = 1$ and $\vec{a}_1$ is orthogonal to $\vec{a}_2$, i.e., $\vec{a}_1 \perp \vec{a}_2$. **Show that $\vec{a}_1$ and $\vec{a}_2$ are linearly independent**.

Hint: Consider a proof by contradiction (assume $\vec{a}_1$ and $\vec{a}_2$ are linearly dependent, i.e., $\vec{a}_1 = \beta\vec{a}_2$ for $\beta \in \mathbb{R}$).

Solution:

Known:

$\mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$ is an orthonormal matrix, i.e., $||\vec{a}_1|| = ||\vec{a}_2|| = 1$ and $\vec{a}_1$ is orthogonal to $\vec{a}_2$.

To show:

$\vec{a}_1$ and $\vec{a}_2$ are linearly independent.

Proof:

Let us prove using the contradiction method. Assume that $\vec{a}_1$ and $\vec{a}_2$ are linearly dependent, i.e., $\vec{a}_1 = \beta\vec{a}_2$ for $\beta \in \mathbb{R}$.

Since $||\vec{a}_1|| = 1$, $||\beta\vec{a}_2|| = |\beta| \cdot ||\vec{a}_2|| = |\beta| = 1$.

Also, $\vec{a}_1$ is orthogonal to $\vec{a}_2$, which means $\langle \vec{a}_1, \vec{a}_2 \rangle = 0$. Hence, $\langle \beta\vec{a}_2, \vec{a}_2 \rangle = \beta \langle \vec{a}_2, \vec{a}_2 \rangle = \beta ||\vec{a}_2||^2 = \beta = 0$.

Thus, we arrive at a contradiction, implying that our initial assumption was wrong. Therefore, $\vec{a}_1$ and $\vec{a}_2$ are linearly independent.

Alternative Proof:

$\vec{a}_1$ and $\vec{a}_2$ are linearly independent if we can show that for $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1\vec{a}_1 + \beta_2\vec{a}_2 = 0$ implies $\beta_1 = \beta_2 = 0$.

$$\begin{aligned} \beta_1\vec{a}_1 + \beta_2\vec{a}_2 &= 0 \\ ||\beta_1\vec{a}_1 + \beta_2\vec{a}_2||^2 &= 0 \\ \beta_1^2||\vec{a}_1||^2 + \beta_2^2||\vec{a}_2||^2 + 2\beta_1\beta_2\langle \vec{a}_1, \vec{a}_2 \rangle &= 0 \\ \beta_1^2 + \beta_2^2 &= 0. \end{aligned}$$

Since the sum of squares of two real numbers is zero, the individual numbers should also be zero, i.e., $\beta_1 = \beta_2 = 0$. Also when $\beta_1 = \beta_2 = 0$, this directly implies that $\beta_1\vec{a}_1 + \beta_2\vec{a}_2 = 0$. Therefore, $\vec{a}_1$ and $\vec{a}_2$ are linearly independent.

(b) (8 points) Now suppose that the matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is such that

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix},$$

where $||\vec{a}_1|| = ||\vec{a}_2|| \cdots = ||\vec{a}_n|| = 1$ and $\vec{a}_1, \cdots, \vec{a}_n$ are pairwise mutually orthogonal, i.e. $\vec{a}_i \perp \vec{a}_j$ for all $i, j = 1, \cdots, n$ and $i \neq j$. For $\vec{b} \in \mathbb{R}^m$, we are given $\langle \vec{a}_i, \vec{b} \rangle = c_i$ for $i = 1, \cdots, n$. **Find the projection of $\vec{b}$ onto Col(A)**, where Col(**A**) represents the column space of **A**. **Write your answer in terms of $\vec{a}_i$ and c_i**. Show your work.

*Hint: The projection of $\vec{b}$ onto Col(**A**) is given by $\mathbf{A}\vec{\hat{x}}$ where $\vec{\hat{x}}$ is the least squares solution of $\mathbf{A}\vec{x} = \vec{b}$.*

Solution: The least squares solution is given by:

$$
\begin{aligned}
\vec{\hat{x}} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \\
&= \left(\begin{bmatrix} - & \vec{a}_1{}^T & - \\ & \vdots & \\ - & \vec{a}_n{}^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \right)^{-1} \begin{bmatrix} - & \vec{a}_1{}^T & - \\ & \vdots & \\ - & \vec{a}_n{}^T & - \end{bmatrix} \vec{b} \\
&= \begin{bmatrix} \vec{a}_1{}^T\vec{a}_1 & \cdots & \vec{a}_1{}^T\vec{a}_n \\ \vdots & \ddots & \vdots \\ \vec{a}_n{}^T\vec{a}_1 & \cdots & \vec{a}_n{}^T\vec{a}_n \end{bmatrix}^{-1} \begin{bmatrix} - & \vec{a}_1{}^T & - \\ & \vdots & \\ - & \vec{a}_n{}^T & - \end{bmatrix} \vec{b} \\
&= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \vec{a}_1{}^T\vec{b} \\ \vdots \\ \vec{a}_n{}^T\vec{b} \end{bmatrix} \\
&= \begin{bmatrix} \langle \vec{a}_1, \vec{b} \rangle \\ \vdots \\ \langle \vec{a}_n, \vec{b} \rangle \end{bmatrix} \\
&= \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.
\end{aligned}
$$

Therefore, the projection is:

$$
\begin{aligned}
\mathbf{A}\vec{\hat{x}} &= \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
&= c_1\vec{a}_1 + \cdots + c_n\vec{a}_n \\
&= \sum_{i=1}^{n} c_i\vec{a}_i.
\end{aligned}
$$

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11. Gold Code Inner Product Circuits (27 points)

Recall that Gold codes are sequences with elements equal to ± 1. We often need to compare the similarity of Gold codes by finding their inner product. In this problem, we will try to design a circuit that can compute inner products of Gold codes.

(a) (2 points) We are given two Gold codes $\vec{s}_1 = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$ and $\vec{s}_2 = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix}^{\mathrm{T}}$ each of length 6. The codes are represented by time-varying voltage signals $V_1(t), V_2(t)$ that map the ± 1 elements to $\pm 1\,\mathrm{V}$ symbols of length $1\,\mathrm{ms}$ as shown in Figure 11.1.

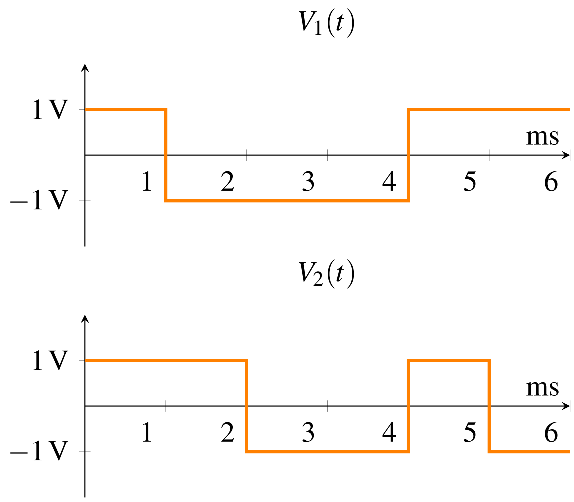


Figure 11.1: Time-varying voltage signals $V_1(t), V_2(t)$ that represent $\vec{s}_1, \vec{s}_2$ respectively

Compute the inner product $\langle \vec{s}_1, \vec{s}_2 \rangle$. Show your work.

Solution:

We can compute

$$\langle \vec{s}_1, \vec{s}_2 \rangle = (1)(1) + (-1)(1) + (-1)(-1) + (-1)(-1) + (1)(1) + (1)(-1) = 2\,.$$

Note that for Gold code signals, this can also be computed as the difference between the number of matching elements and non-matching elements:

$$\langle \vec{s}_1, \vec{s}_2 \rangle = n_{\text{matching elements}} - n_{\text{non-matching elements}} = 4 - 2 = 2\,.$$

(b) (8 points) For Gold code signals, we notice that the multiplication of ± 1 elements is equivalent to checking if the two elements are equal. In the inner product circuit shown in Figure 11.2, an inverting summer and a match detect circuit are used to check when $V_1(t) = V_2(t)$. The results from the match detect circuit are then integrated across the length of the signal to produce V_{IP} which represents the final inner product value.

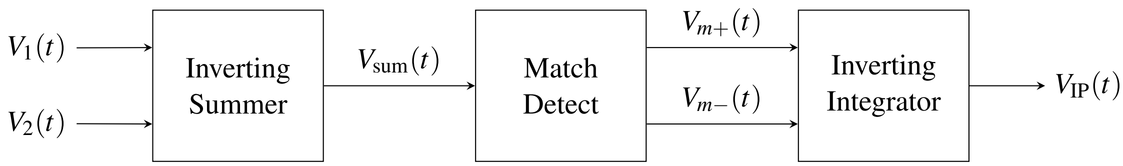


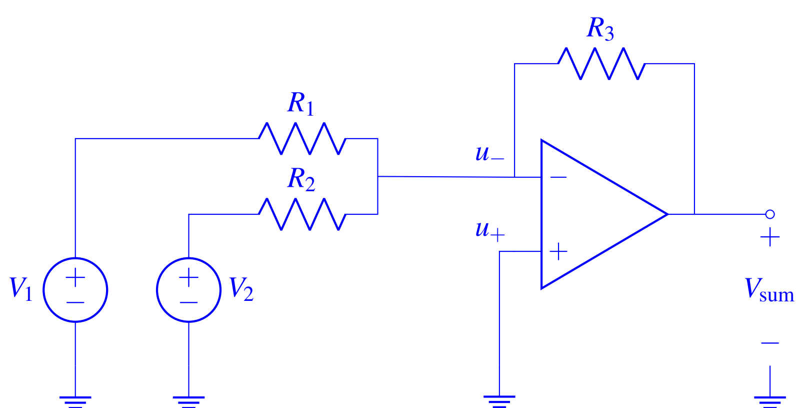
Figure 11.2: Block diagram of inner product circuit

In this part, we wish to design the inverting summer block. We have access to a single ideal op-amp (already drawn) and up to three resistors for which we can choose values. No other components are available. **Design a circuit such that $V_{\text{sum}} = -V_1 - V_2$. Label the resistances for all resistors used.**

Solution:

Note: the inverting summing amplifier is analyzed in detail in Note 19.

The inverting summing amplifier can be designed as follows:



We can analyze this circuit using superposition. By the op-amp golden rules, we know that $u_- = u_+ = 0\,\text{V}$. Thus if we apply superposition and keep V_1 on and turn V_2 off, we notice that both sides of R_2 are $0\,\text{V}$ and no current will flow through. In other words, we are just left with the inverting amplifier configuration where $V_{\text{sum},1} = -\frac{R_3}{R_1}V_1$. Repeating the process by turning V_2 on and V_1 off gives $V_{\text{sum},2} = -\frac{R_3}{R_2}V_2$. Adding these two results together yields

$$V_{\text{sum}} = -\frac{R_3}{R_1}V_1 - \frac{R_3}{R_2}V_2\,.$$

For this problem, we want $\frac{R_3}{R_1} = \frac{R_3}{R_2} = 1$, which means that we must have $R_1 = R_2 = R_3$. For example, we can choose $R_1 = R_2 = R_3 = 1\,\text{k}\Omega$, but any solution with the inverting summing amplifier and equal value resistors is valid.

(c) (5 points) The match detect circuit wants to use $V_{\text{sum}} = -V_1 - V_2$ to determine when $V_1 = V_2$. When $V_{\text{sum}} = 2\,\text{V}$, we know that V_1 and V_2 match with value $-1\,\text{V}$. When $V_{\text{sum}} = -2\,\text{V}$, we know that V_1 and V_2 match with value $1\,\text{V}$. The match detect circuit can be implemented using comparators with outputs V_{m+} and V_{m-} as shown in Figure 11.4.

V_{sum}	V_{m+}	V_{m-}
$2\,\text{V}$	$1\,\text{V}$	$-3\,\text{V}$
$0\,\text{V}$	$1\,\text{V}$	$1\,\text{V}$
$-2\,\text{V}$	$-3\,\text{V}$	$1\,\text{V}$

Figure 11.3: Input output table

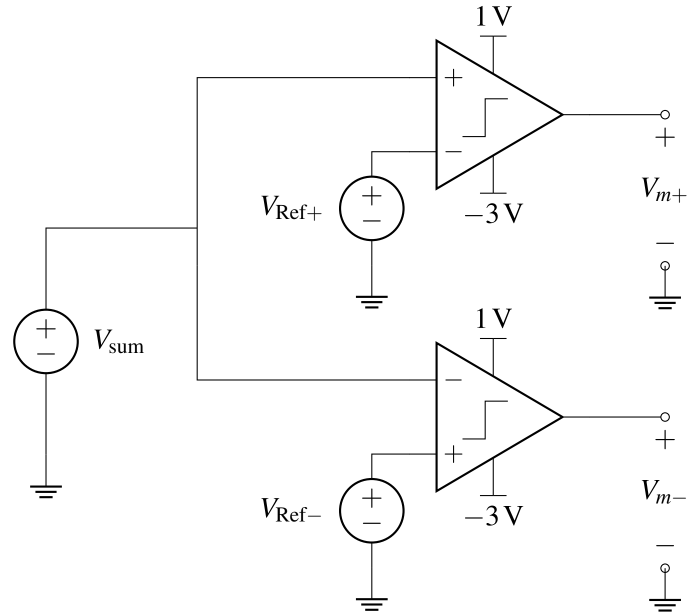


Figure 11.4: Match detect circuit

Choose values for $V_{\text{Ref}+}$ and $V_{\text{Ref}-}$ such that $V_{\text{sum}}, V_{m+}, V_{m-}$ satisfy the table in Figure 11.3. Justify your answer.

Solution:

For the comparator that outputs V_{m+}, we know that $V_{m+} = 1\,\text{V}$ when $V_{\text{sum}} > V_{\text{Ref}+}$ and $V_{m+} = -3\,\text{V}$ when $V_{\text{sum}} < V_{\text{Ref}+}$. We can use the input output table values to find the restrictions on $V_{\text{Ref}+}$.

$$\begin{aligned} V_{\text{sum}} = 2\,\text{V}, V_{m+} = 1\,\text{V} &\longrightarrow 2\,\text{V} > V_{\text{Ref}+} \\ V_{\text{sum}} = 0\,\text{V}, V_{m+} = 1\,\text{V} &\longrightarrow 0\,\text{V} > V_{\text{Ref}+} \\ V_{\text{sum}} = -2\,\text{V}, V_{m+} = -3\,\text{V} &\longrightarrow -2\,\text{V} < V_{\text{Ref}+}\,, \end{aligned}$$

which means that $-2\,\text{V} < V_{\text{Ref}+} < 0\,\text{V}$. Similarly for V_{m-}, $V_{m-} = 1\,\text{V}$ when $V_{\text{Ref}-} > V_{\text{sum}}$ and $V_{m-} = -3\,\text{V}$ when $V_{\text{Ref}-} < V_{\text{sum}}$. Again, the table values give

$$\begin{aligned} V_{\text{sum}} = 2\,\text{V}, V_{m-} = -3\,\text{V} &\longrightarrow V_{\text{Ref}-} < 2\,\text{V} \\ V_{\text{sum}} = 0\,\text{V}, V_{m-} = 1\,\text{V} &\longrightarrow V_{\text{Ref}-} > 0\,\text{V} \\ V_{\text{sum}} = -2\,\text{V}, V_{m-} = 1\,\text{V} &\longrightarrow V_{\text{Ref}-} > -2\,\text{V}\,, \end{aligned}$$

which means that $0\,\text{V} < V_{\text{Ref}-} < 2\,\text{V}$. Any choices of $V_{\text{Ref}+}$ and $V_{\text{Ref}-}$ in these ranges are valid, for example $V_{\text{Ref}+} = -1\,\text{V}$ and $V_{\text{Ref}-} = 1\,\text{V}$.

(d) (7 points) In order to compute the inner product, we can use the circuit in Figure 11.5 to integrate the match signals. You may assume the op-amp is ideal.

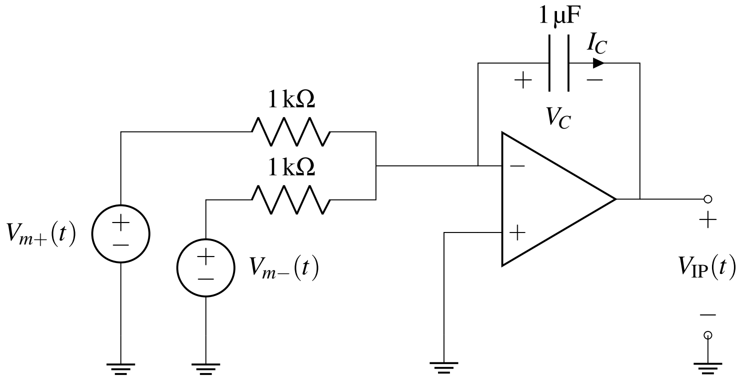


Figure 11.5: Inverting integrator circuit

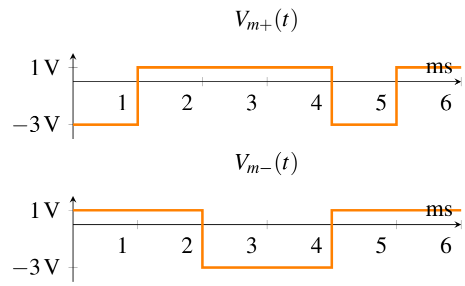


Figure 11.6

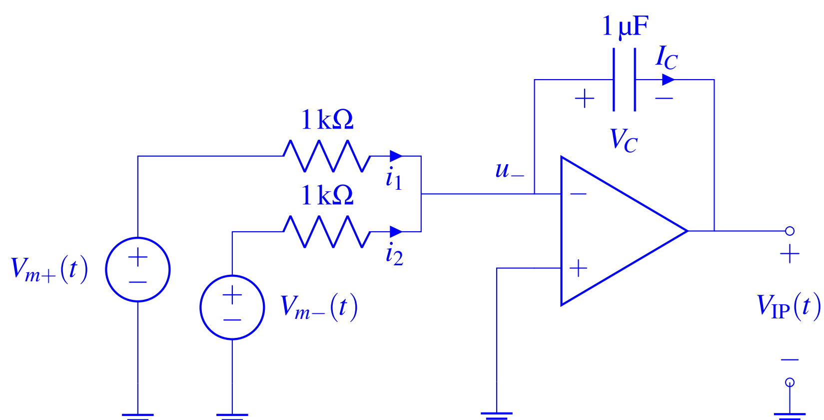
The waveforms for $V_{m+}(t)$ and $V_{m-}(t)$ are given in Figure 11.6.

- **Plot $I_C(t)$ from $t = 0\,\text{ms}$ to $t = 6\,\text{ms}$ in the graph provided. Label the units and current values in your graph.**
- **Compute $V_{\text{IP}}(6\,\text{ms})$. Assume that $V_{\text{IP}}(0) = 0\,\text{V}$.**

Show your work.

Solution:

The analysis for this circuit is similar to the inverting summer.



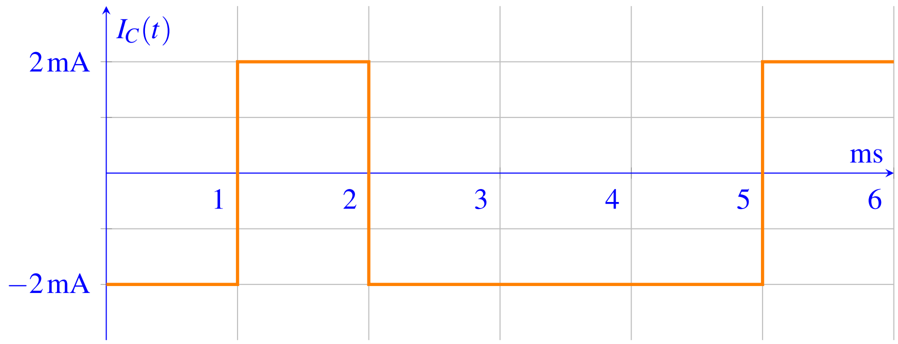
According to the op-amp golden rules, we know that $u_- = u_+ = 0$ and thus we can compute

$$i_1 = \frac{V_{m+}}{1\,\text{k}\Omega}, \quad i_2 = \frac{V_{m-}}{1\,\text{k}\Omega}.$$

Using KCL at the u_- node, we see that $I_C = i_1 + i_2$, and thus

$$I_C = \frac{V_{m+} + V_{m-}}{1\,\text{k}\Omega}.$$

Using the given waveforms for $V_{m+}(t)$ and $V_{m-}(t)$, we can compute and graph $I_C(t)$ as follows:



We can now compute $V_{\text{IP}}(6\,\text{ms})$ since we know that $I_C = 1\,\mu\text{F}\frac{\text{d}V_C}{\text{d}t}$ and $V_C = -V_{\text{IP}}$. Integrating both sides and rearranging yields

$$V_{\text{IP}}(6\,\text{ms}) - V_{\text{IP}}(0) = -\int_0^{6\,\text{ms}} \frac{I_C(t)}{1\,\mu\text{F}} dt\,,$$

which in this case can be done by computing the area under the curve in our graph for $I_C(t)$. Since we are given $V_{\text{IP}}(0) = 0\,\text{V}$, we now have

$$V_{\text{IP}}(6\,\text{ms}) = -\frac{1}{1\,\mu\text{F}}\int_0^{6\,\text{ms}} I_C(t)dt = -\frac{1}{1\,\mu\text{F}}(-2+2-2-2-2+2)\text{mA}\cdot\text{ms} = 4\,\text{V}\,.$$

(e) (5 points) As we increase the signal length, the maximum/minimum value of V_{IP} also increases. In order to keep the output voltage to a manageable level, we decide to switch out the $1\,\mu\text{F}$ capacitor for a variable capacitor shown in Figure 11.7.

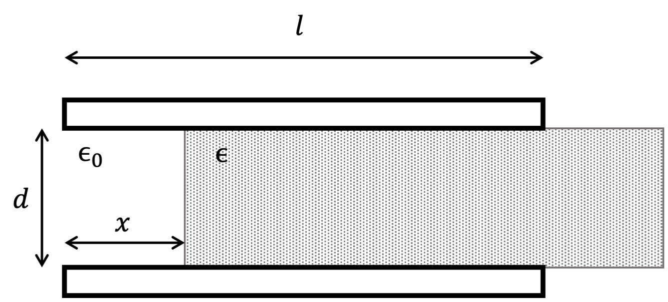


Figure 11.7: Variable capacitor

The capacitor has square plates with **length and width l with a separation of d**. Inside, we have a dielectric material with **permittivity $\varepsilon = 5\varepsilon_0$** that we can slide to change the total capacitance between the plates. x measures the displacement of the dielectric material. Assuming $0 \leq x \leq l$, **find the total capacitance C in terms of l, d, x, ε_0**. You do not need to reduce your answer. Show your work.

Solution:

The variable capacitor can be split into two sections, one with the dieletric material and one with just air. The capacitance of the section with air has length x, width l and separation d, and can be computed as

$$C_1 = \varepsilon_0 \frac{lx}{d}\,.$$

The capacitance of the section with dielectric material has length $l - x$, width l and separation d, and can be computed as

$$C_2 = 5\varepsilon_0 \frac{l(l-x)}{d}\,.$$

Since these capacitors are in parallel, the total capacitance can be found as

$$C = C_1 + C_2 = \varepsilon_0 \frac{lx}{d} + 5\varepsilon_0 \frac{l(l-x)}{d}.$$