EECS 16A Designing Information Devices and Systems I
Fall 2023 Midterm 1

Midterm 1 Solution

PRINT your student ID: ____________________

PRINT AND SIGN your name: ____________, ____________ ____________
(last name) (first name) (signature)

PRINT your discussion section and GSI: ____________________

Name and SID of the person to your left: ____________________

Name and SID of the person to your right: ____________________

Name and SID of the person in front of you: ____________________

Name and SID of the person behind you: ____________________

1. **Honor Code (0 Points)**
Acknowledge that you have read and agree to the following statement and sign your name below:
As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.
If you do not sign your name, you will get a 0 on the exam.

2. **When the exam starts, write your SID at the top of every page. (2 Points)**
No extra time will be given for this task.

3. **Tell us about something you did in the last year that you are proud of. (1 Point)**
Any answer, as long as you write it down, will be given full credit.

4. **What is a movie you watched recently that made you happy? (1 Point)**
Any answer, as long as you write it down, will be given full credit.

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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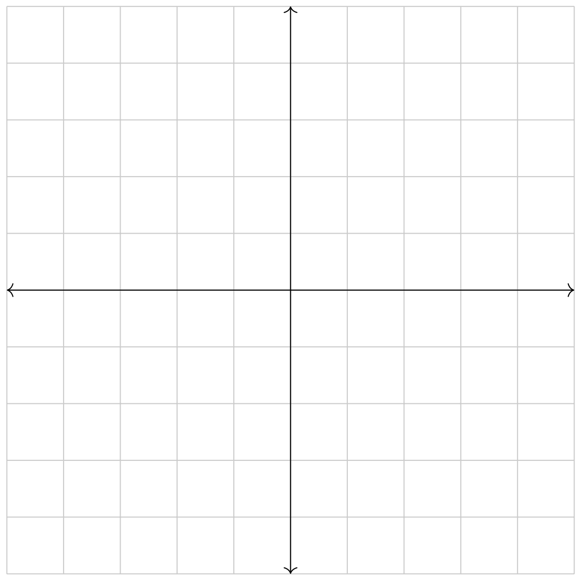
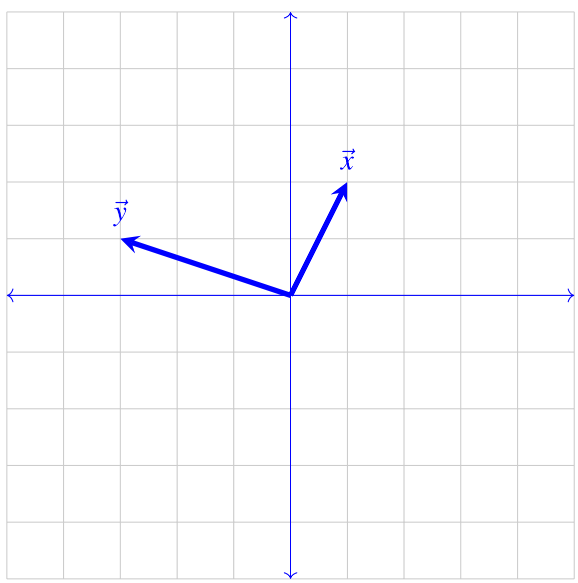
5. Transform that Vector! (8 points)

(a) (2 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. **Graph the vectors $\vec{x}$ and $\vec{y}$ and label them**.

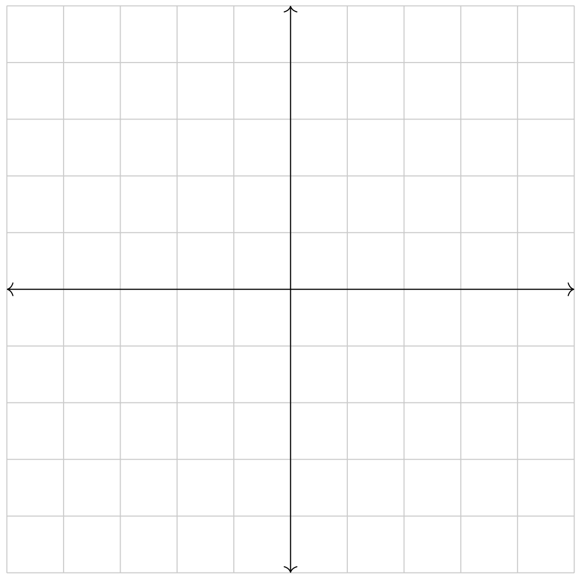


Solution:



(b) (3 points) Now apply a transformation matrix $\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ to get new vectors $\vec{v} = \mathbf{T}\vec{x}$ and $\vec{w} = \mathbf{T}\vec{y}$.

Graph the vectors $\vec{v}$ and $\vec{w}$ and label them.

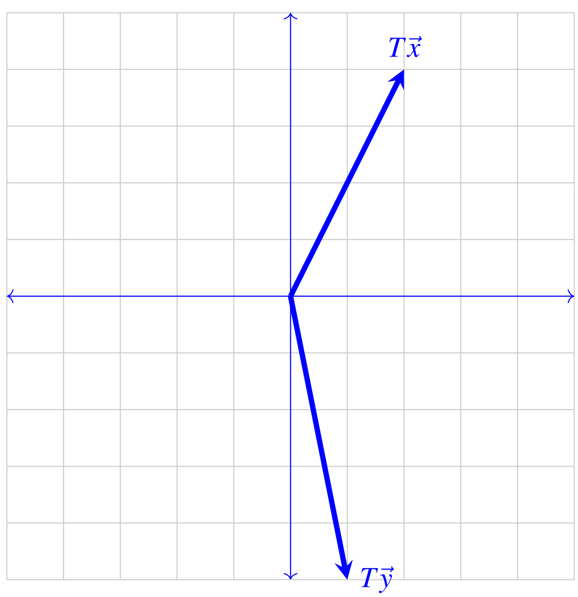


Solution:

We can compute $\vec{v}, \vec{w}$ as follows:

$$\vec{v} = \mathbf{T}\vec{x} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\vec{w} = \mathbf{T}\vec{y} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$



(c) (3 points) Let $\vec{x} \in \mathbb{R}^2$. You first transform $\vec{x}$ by $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and then by $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. **Write the entries of the matrix $\mathbf{C} \in \mathbb{R}^{2\times 2}$ that represents the combined transformation**. Show your work.

Solution:

As we saw in the previous part, we can apply a transformation matrix to a vector by multiplying from the left. Thus if first apply $\mathbf{A}$ and then $\mathbf{B}$, we get

$$\mathbf{C} = \mathbf{BA} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Note that due to the order of the transformation applied, $\mathbf{C} \neq \mathbf{AB}$ since matrix multiplication is not commutative.

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6. Shiny New Monitor (12 points)

You are trying to calibrate your new computer monitor. For this you ask the monitor to generate one color for all of its pixels, represented by $\vec{x}_{\text{disp}} = \begin{bmatrix} r \\ g \\ b \end{bmatrix}$, where r, g, b represent the intensities of red, green, and blue light output by the monitor.

You are calibrating using sensors that measure linear combinations of red, green, and blue light. For sensor i, the sensor measurement m_i is

$$m_i = s_{ir}r + s_{ig}g + s_{ib}b \tag{1}$$

where r, g, b are the intensities of red, green, and blue light output by the monitor, and s_{ir}, s_{ig}, s_{ib} are the sensitivities to red, green, and blue respectively for that sensor. The sensitivities of your three sensors are given by:

	Sensitivity to red (s_{ir})	Sensitivity to green (s_{ig})	Sensitivity to blue (s_{ib})
Sensor 1	0.75	0	0.5
Sensor 2	1	1	1
Sensor 3	1	0.5	0

(a) (4 points) You run the calibration and collect the following data:

	Sensor measurement (m_i)
Sensor 1	380
Sensor 2	420
Sensor 3	220

Write a matrix vector equation that allows you to compute $\vec{x}_{\text{disp}}$. You do not have to solve this equation.

Solution:

We can write three linear equations corresponding to each sensor as follows:

$$\begin{aligned} 0.75r + 0.5b &= 380 \\ r + g + b &= 420 \\ r + 0.5g &= 220\,. \end{aligned}$$

This system of linear equations can be represented in matrix-vector form:

$$\begin{bmatrix} 0.75 & 0 & 0.5 \\ 1 & 1 & 1 \\ 1 & 0.5 & 0 \end{bmatrix} \vec{x}_{\text{disp}} = \begin{bmatrix} 380 \\ 420 \\ 220 \end{bmatrix}.$$

(b) (5 points) Suppose the sensor data for a different measurement gives you the matrix equation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 1 & 1 \\ 0 & 0.5 & 0.5 \end{bmatrix} \vec{x}_{\text{disp}} = \begin{bmatrix} 420 \\ 360 \\ 120 \end{bmatrix}. \tag{2}$$

Use Gaussian elimination to find the vector $\vec{x}_{\text{disp}}$ that satisfies the equation. If there is a unique solution, state it. If there is no solution, explain why. If there are infinite solutions, parameterize your solution. Show your work.

Solution:

We begin by writing our matrix equation in augmented form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 0.5 & 1 & 1 & 360 \\ 0 & 0.5 & 0.5 & 120 \end{array}\right].$$

We can now use Gaussian elimination to solve:

$$\begin{aligned}
&\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 0.5 & 1 & 1 & 360 \\ 0 & 0.5 & 0.5 & 120 \end{array}\right] \\
\longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 1 & 2 & 2 & 720 \\ 0 & 1 & 1 & 240 \end{array}\right] && R_2 \leftarrow 2R_2,\ R_3 \leftarrow 2R_3 \\
\longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 0 & 1 & 2 & 300 \\ 0 & 1 & 1 & 240 \end{array}\right] && R_2 \leftarrow R_2 - R_1 \\
\longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 0 & 1 & 2 & 300 \\ 0 & 0 & -1 & -60 \end{array}\right] && R_3 \leftarrow R_3 - R_2 \\
\longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 0 & 1 & 2 & 300 \\ 0 & 0 & 1 & 60 \end{array}\right] && R_3 \leftarrow -R_3 \\
\longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 420 \\ 0 & 1 & 0 & 180 \\ 0 & 0 & 1 & 60 \end{array}\right] && R_2 \leftarrow R_2 - 2R_3 \\
\longrightarrow &\left[\begin{array}{ccc|c} 1 & 0 & 0 & 240 \\ 0 & 1 & 0 & 180 \\ 0 & 0 & 1 & 60 \end{array}\right] && R_1 \leftarrow R_1 - R_2\,.
\end{aligned}$$

Since we have no inconsistencies and every column has a pivot, there exists a unique solution! The solution is given by

$$\vec{x}_{disp} = \begin{bmatrix} 240 \\ 180 \\ 60 \end{bmatrix}.$$

(c) (3 points) You are interested in developing a new display technology that includes a fourth white channel (w). The white channel can be expressed as a linear combination of all the other channels as shown below:

$$w = \frac{1}{3} \cdot r + \frac{1}{3} \cdot g + \frac{1}{3} \cdot b. \tag{3}$$

Find a transformation matrix T that transforms the vector $\begin{bmatrix} r \\ g \\ b \end{bmatrix}$ to the vector $\begin{bmatrix} w \\ r \\ g \\ b \end{bmatrix}$.

$$\mathbf{T}\begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} w \\ r \\ g \\ b \end{bmatrix}.$$

Solution: Using equation (3), we can rewrite the desired transformation as:

$$\mathbf{T}\begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{3}r + \frac{1}{3}g + \frac{1}{3}b \\ r \\ g \\ b \end{bmatrix}. \tag{4}$$

In order for equation (4) to be valid, we notice that $\mathbf{T} \in \mathbb{R}^{4\times 3}$. Since the first column of $\mathbf{T}$ represents the coefficients for r, the second column for g, and the third for b, we can populate the matrix as follows:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} w \\ r \\ g \\ b \end{bmatrix}.$$

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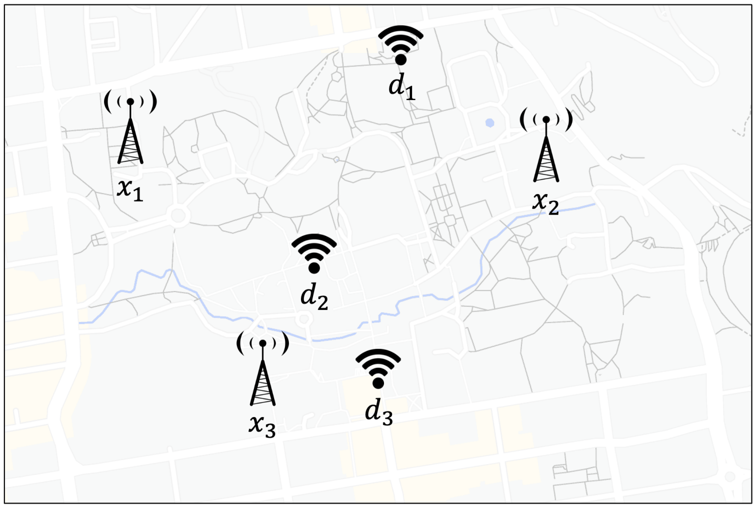
7. Campus WiFi (13 points)

EECS16A is tasked with troubleshooting campus WiFi connectivity issues.

The campus has three WiFi **transmitters**. Let $x_1, x_2, x_3 \in \mathbb{R}$ be the signal strength generated by each transmitter. In addition, we set up three WiFi **detectors** across campus that measure values $d_1, d_2, d_3 \in \mathbb{R}$ respectively.



The transmitted signal strengths and measured detector values are represented in a transmitter vector $\vec{x}$ and detector vector $\vec{d}$ as follows

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

After careful study, we have found that $\vec{x}$ and $\vec{d}$ are related by the signal mapping matrix $\mathbf{M}$ as shown below:

$$\mathbf{M}\vec{x} = \vec{d}. \tag{5}$$

(a) (4 points) For this part consider

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

What is the dimension of the column space of M? Justify your answer.

Solution:

Here, we see that $2 \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and thus the first two columns are linearly dependent. As a result, we have

$$\mathrm{Col}(\mathbf{M}) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right).$$

Since the remaining column vectors are linearly dependent, they represent a basis for our column space. The dimension of the column space is equal to the number of basis vectors, which in this case is 2.

(b) (4 points) Due to campus budget cuts, we are only allowed to use two detectors. Our transmitter and detector vectors are now

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

For this part we are given a new mapping matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}.$$

We notice that certain combinations of transmitter signal strengths produce interference that results in no signal at our detectors. We suspect this is connected to the null space. **Find a basis for the null space of M.** Show your work.

Solution:

In order to solve for the null space, we are looking for vectors $\vec{x}$ such that $\mathbf{M}\vec{x} = \vec{0}$. Let's set this up as an augmented matrix and use Gaussian elimination!

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right] \\ \longrightarrow &\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \qquad\qquad R_2 \leftarrow R_2 - R_1\,. \end{aligned}$$

Since we have a column with no pivot (column 3), we know we have infinite solutions. Let the variable corresponding to the pivotless column $x_3 = t$ be the free variable. Now the two rows read

$$\begin{aligned} x_1 - 2t = 0 &\longrightarrow x_1 = 2t \\ x_2 - t = 0 &\longrightarrow x_2 = t \end{aligned}$$

which gives us the solution

$$\vec{x} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} t\,.$$

Thus the basis for this null space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Note that any scalar multiple of the given vector is also a valid basis vector!

(c) (5 points) For this part consider a new signal mapping matrix **M** which has null space

$$\text{null}(\mathbf{M}) = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

We find that we can achieve $\vec{d} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ with the following transmitter vector:

$$\mathbf{M}\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Unfortunately having large values in the transmitter vector $\vec{x}$ make it expensive to operate! UC Berkeley tells us they can only set the signal strength to an integer value from 0 to 5. **Can you find a transmitter vector $\vec{x}$ such that $\mathbf{M}\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ where $x_1, x_2, x_3 \in \{0,1,2,3,4,5\}$? If yes, state the vector and show your work. If not justify why no such vector exists.**

Solution:

Given an equation $\mathbf{M}\vec{x} = \vec{d}$, a null space vector $\vec{v}$, and scalar $\alpha \in \mathbb{R}$, we know that:

$$\mathbf{M}\vec{x} = \vec{d} \text{ and } \mathbf{M}\alpha\vec{v} = \vec{0}.$$

Adding these two equations together, we see

$$\mathbf{M}\vec{x} + \mathbf{M}\alpha\vec{v} = \vec{d} + \vec{0}$$
$$\mathbf{M}(\vec{x} + \alpha\vec{v}) = \vec{d}.$$

For our problem, this means that

$$\mathbf{M}\left(\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

We can now choose a value for α such that our signal strength vector is in the desired range! In this case, we choose $\alpha = -1$ so that

$$\mathbf{M}\left(\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$
$$\mathbf{M}\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix},$$

which means the desired transmitter vector is

$$\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

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8. Observing Cars (10 points)

Consider an autonomous car with state vector $\vec{x}[t] \in \mathbb{R}^n$ at time t where

$$\vec{x}[t] = \begin{bmatrix} x_1[t] \\ x_2[t] \\ \vdots \\ x_n[t] \end{bmatrix}.$$

$\vec{x}[t]$ updates according to the following equation for every timestep t:

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t]\,, \mathbf{A} \in \mathbb{R}^{n\times n}\,. \tag{6}$$

The state of the car is observed through LIDAR measurements $\vec{y}[t] \in \mathbb{R}^m$, which are related to the state of the car for every timestep t as follows:

$$\vec{y}[t] = \mathbf{C}\vec{x}[t]\,, \mathbf{C} \in \mathbb{R}^{m\times n}\,. \tag{7}$$

In this question, we are concerned with the design of the matrix $\mathbf{C}$.

(a) (3 points) **Express $\vec{y}[t+1]$ in terms of $\vec{x}[t]$, $\mathbf{A}$, and $\mathbf{C}$.** Show your work.

Solution:

We are given that $\vec{y}[t] = \mathbf{C}\vec{x}[t]$, which we can reindex to get $\vec{y}[t+1] = \mathbf{C}\vec{x}[t+1]$. Now we can substiute the given equation $\vec{x}[t+1] = \mathbf{A}\vec{x}[t]$ to get

$$\begin{aligned} \vec{y}[t+1] &= \mathbf{C}\vec{x}[t+1] \\ &= \mathbf{C}\mathbf{A}\vec{x}[t]\,. \end{aligned}$$

(b) (7 points) Let $n > 5$. Assume that the first 5 columns of $\mathbf{A}$ are in the null space of $\mathbf{C}$ and no other columns of $\mathbf{A}$ are in the null space of $\mathbf{C}$. **Write $\vec{y}[t+1]$ in terms of $\mathbf{C}$, $x_1[t], \ldots, x_n[t]$, and the columns of $\mathbf{A}$. Which entries of $\vec{x}[t]$ play no role in the value of $\vec{y}[t+1]$?** Justify your answer and show your work.

Solution:

Since we are dealing with the columns of $\mathbf{A}$, lets label the column vectors as $\vec{a}_1, \ldots, \vec{a}_n$. Recall that we can represent matrix-vector multiplication as a linear combination of the columns of the matrix. Thus we can rewrite the answer of the previous part as

$$\begin{aligned} \vec{y}[t+1] &= \mathbf{C}\mathbf{A}\vec{x}[t] \\ &= \mathbf{C}\left(\mathbf{A}\vec{x}[t]\right) \\ &= \mathbf{C}\left(\sum_{i=1}^{n} \vec{a}_i x_i[t]\right). \end{aligned}$$

Now we can distribute the left multiplication of $\mathbf{C}$ which gives

$$\begin{aligned} \vec{y}[t+1] &= \mathbf{C}\left(\sum_{i=1}^{n} \vec{a}_i x_i[t]\right) \\ &= \sum_{i=1}^{n} \mathbf{C}\vec{a}_i x_i[t]\,. \end{aligned}$$

We are given that the first five columns of $\mathbf{A}$ are in the null space of $\mathbf{C}$. Thus $\mathbf{C}\vec{a}_i = \vec{0}$ for $i = 1, \ldots, 5$, which means that the first five terms in the sum above simplify to the zero vector.

$$\begin{aligned}\vec{y}[t+1] &= \sum_{i=1}^{n} \mathbf{C}\vec{a}_i x_i[t] \\ &= \sum_{i=1}^{5} \mathbf{C}\vec{a}_i x_i[t] + \sum_{i=6}^{n} \mathbf{C}\vec{a}_i x_i[t] \\ &= \sum_{i=6}^{n} \mathbf{C}\vec{a}_i x_i[t]\,.\end{aligned}$$

We have found that the first five elements of $\vec{x}[t]$, namely $x_1[t], \cdots, x_5[t]$, play no role in the value of $\vec{y}[t+1]$.

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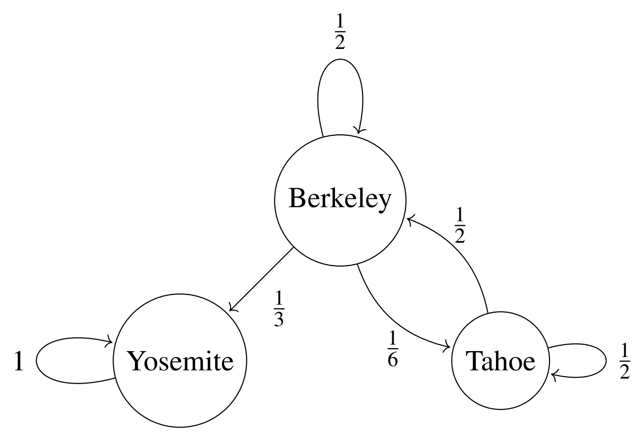
9. Migration of the Bears (23 points)

A population of black bears migrate between Berkeley, Yosemite, and Tahoe every year. The population of black bears in each location is represented in a state vector $\vec{b}[i]$ defined as

$$\vec{b}[i] = \begin{bmatrix} \text{number of black bears in Berkeley in year } i \\ \text{number of black bears in Yosemite in year } i \\ \text{number of black bears in Tahoe in year } i \end{bmatrix}.$$

The migration of the black bears follows the state transition system modeled by the following diagram:



(a) (4 points) **Find the state transition matrix for this system T such that $\vec{b}[i+1] = \mathbf{T}\vec{b}[i]$ and state whether the system is conservative.** Justify your answer.

Solution:

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} \end{bmatrix}.$$

The system is conservative because all columns sum to 1.

(b) (5 points) Due to climate change, we find that the migration patterns of the bears have changed and the new state transition matrix $\mathbf{S}$ such that $\vec{b}[i+1] = \mathbf{S}\vec{b}[i]$ is as follows:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 \end{bmatrix}.$$

Find the matrix M such that $\vec{b}[i-1] = \mathbf{M}\vec{b}[i]$. Show your work.

Solution:
We note that $\mathbf{M} = \mathbf{S}^{-1}$. We can find the inverse of $\mathbf{S}$ using Gaussian elimination as follows:

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1 \times 2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 & -1 & 0 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - \frac{R_1}{2}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & \frac{2}{3} & 1 & -1 & 0 & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 \times 3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - \frac{2}{3}R_2$$

So,

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -1 & -2 & 1 \end{bmatrix}.$$

(c) (6 points) Berkeley researchers discover a new species of bear, the Oski bears (Ursus Oskius), which migrate between the three locations according to the transition matrix:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

This system has a steady state. Compute the eigenspace corresponding to the steady state. Show your work.

Solution:
We are told that $\mathbf{P}$ has a steady state which means it has an eigenvalue of 1. Recall that if $\vec{v}$ is an eigenvector of a matrix A corresponding to the eigenvalue λ, $A\vec{v} = \lambda\vec{v}$. Hence, $(A - \lambda I)\vec{v} = \vec{0}$. Let's

solve for the case where $\lambda = 1$:

$$\left[\begin{array}{ccc|c} \frac{1}{2}-1 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3}-1 & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3}-1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right] \qquad R_1 \leftarrow R_1 \times (-2)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right] \qquad R_3 \leftarrow R_3 - \frac{R_1}{2}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right] \qquad R_2 \leftarrow R_2 \times (-3)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 \leftarrow R_3 - \left(\frac{1}{3}\right) R_2$$

Since the third column has no pivot, let $v_3 = t \in \mathbb{R}$ be our free variable. Now we can solve for v_1, v_2 as follows:

$$\begin{aligned} v_1 - \frac{2}{3}v_3 = 0 \quad &\longrightarrow \quad v_1 = \frac{2}{3}t \\ v_2 - v_3 = 0 \quad &\longrightarrow \quad v_2 = t\,. \end{aligned}$$

Writing the above in vector forms gives

$$\vec{v} = \begin{bmatrix} \frac{2}{3}t \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix} t$$

which means the steady state eigenvector is $\begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$. Thus the steady state eigenspace is

$$\text{span}\left(\begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}\right).$$

Note that the span of any scalar multiple of the vector given above is a valid eigenspace!

(d) (4 points) As part of your undergraduate research project, you are asked to simulate the migration of the Oski bears. Suppose you are given a new transition matrix $\mathbf{Q}$ and an initial state

$$\vec{b}[0] = 108\vec{v}_1 + 36\vec{v}_2\,,$$

where $\vec{v}_1$ and $\vec{v}_2$ denote the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{5}{6}$ respectively for the matrix $\mathbf{Q}$. **Find the state vector after two timesteps $\vec{b}[2]$ in terms of $\vec{v}_1$ and $\vec{v}_2$.** Show your work.

Solution:
For an eigenvector $\vec{v}$ of a matrix $\mathbf{Q}$ corresponding to the eigenvalue λ, $\mathbf{Q}\vec{v} = \lambda v \implies \mathbf{Q}^2\vec{v} = \lambda^2\vec{v}$. Using this, we can write

$$\begin{aligned}
\vec{b}[2] &= \mathbf{Q}\vec{b}[1] \\
&= \mathbf{Q}(\mathbf{Q}\vec{b}[0]) \\
&= \mathbf{Q}^2(108\vec{v}_1 + 36\vec{v}_2) \\
&= 108\mathbf{Q}^2\vec{v}_1 + 36\mathbf{Q}^2\vec{v}_2 \\
&= 108\lambda_1^2\vec{v}_1 + 36\lambda_2^2\vec{v}_2 \\
&= 108 \times 1^2 \times \vec{v}_1 + 36 \times \left(\frac{5}{6}\right)^2 \times \vec{v}_2 \\
&= 108\vec{v}_1 + 25\vec{v}_2.
\end{aligned}$$

(e) (4 points) We are also interested in the wild fish population in Yosemite and Tahoe, represented by $\vec{a}[i]$, as they are an important source of food for the bears. The fish population changes according to $\vec{a}[i+1] = \mathbf{R}\vec{a}[i]$ where

$$\mathbf{R} = \begin{bmatrix} \frac{2}{5} & \frac{1}{3} \\ \frac{3}{5} & \frac{2}{3} \end{bmatrix}.$$

Compute the eigenvalues of R. Does R have a steady state? Justify your answer and show your work.

Solution:
To compute for the eigenvalues of $\mathbf{R}$, we must solve for $\det(\mathbf{R} - \lambda\mathbf{I}) = 0$. In this case,

$$\mathbf{R} - \lambda\mathbf{I} = \begin{bmatrix} \frac{2}{5} - \lambda & \frac{1}{3} \\ \frac{3}{5} & \frac{2}{3} - \lambda \end{bmatrix}.$$

Now we can compute

$$\begin{aligned}
\det(\mathbf{R} - \lambda\mathbf{I}) &= \left(\frac{2}{5} - \lambda\right)\left(\frac{2}{3} - \lambda\right) - \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) \\
&= \frac{2}{15} - \frac{2}{3}\lambda - \frac{2}{5}\lambda + \lambda^2 - \frac{1}{5} \\
&= \lambda^2 - \frac{16}{15}\lambda - \frac{1}{15} \\
&= (\lambda - 1)\left(\lambda - \frac{1}{15}\right).
\end{aligned}$$

The equation

$$\det(\mathbf{R} - \lambda\mathbf{I}) = (\lambda - 1)\left(\lambda - \frac{1}{15}\right) = 0$$

has two solutions where $\lambda = 1$ and $\lambda = \frac{1}{15}$. Since we have an eigenvalue of 1, there exists a steady state!

PRINT your name and student ID: ______________________________

10. Proof (10 points)

Prove the following statement:
If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a **linearly independent set**, then $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \cdots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n\}$ is also a **linearly independent** set.

Hint 1: If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are linearly independent, then $\sum_{i=1}^{n} \alpha_i \vec{v}_i = \vec{0}$ if and only if $\alpha_i = 0$ for all i.

Hint 2: Try proving this first for $n = 3$. Partial credit will be awarded for a valid proof for $n = 3$.

Solution:

Given:
We are given that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ are linearly independent. Using the definition of linear independence given in the hint, this means that

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0} \quad \Longrightarrow \quad \alpha_1 = \cdots = \alpha_n = 0 \,. \tag{8}$$

Want to show:
We want to show that $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \ldots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n\}$ are linearly independent. Using the same defintion of linear independence, this is equivalent to showing that the equation

$$\beta_1(\vec{v}_1 + \vec{v}_2) + \beta_2(\vec{v}_2 + \vec{v}_3) + \cdots + \beta_{n-1}(\vec{v}_{n-1} + \vec{v}_n) + \beta_n \vec{v}_n = \vec{0} \tag{9}$$

is only true when $\beta_i = 0$ for all i.

Proof:
Let us start with the equation (9) and try to prove that $\beta_1 = \beta_2 = \cdots = \beta_n = 0$. We begin by regrouping the terms in equation (9) as follows:

$$\beta_1 \vec{v}_1 + (\beta_1 + \beta_2)\vec{v}_2 + (\beta_2 + \beta_3)\vec{v}_3 + \cdots + (\beta_{n-2} + \beta_{n-1})\vec{v}_{n-1} + (\beta_{n-1} + \beta_n)\vec{v}_n = \vec{0} \,. \tag{10}$$

Notice that the equation above is just a linear combination of $\vec{v}_1, \cdots, \vec{v}_n$ equal to the zero vector. According to our given statement in equation (8), this means that all the coefficients in equation (10) must be zero. In other words, we have

$$\begin{aligned} \beta_1 &= 0 \\ \beta_1 + \beta_2 &= 0 \\ \beta_2 + \beta_3 &= 0 \\ &\vdots \\ \beta_{n-1} + \beta_n &= 0 \,. \end{aligned}$$

Notice that we have n equations and n unknowns, and we can solve for the unique solution by substitution. Namely, from the first equation, we have $\beta_1 = 0$ which we can substiute into the second equation to get $\beta_2 = 0$. Now subsitute $\beta_2 = 0$ into the third equation to get $\beta_3 = 0$. We can repeat this substitution process for all equations which yields the unique solution $\beta_1 = \beta_2 = \cdots = \beta_n = 0$. Thus we have shown that given equation (9), we must have $\beta_i = 0$ for all i and thus $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \ldots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n\}$ is a linearly independent set.