EECS 16A Designing Information Devices and Systems I
Spring 2021 Final Exam – SOLUTIONS

1. (22 points) Linear Algebra

(a) (4 points) Use Gaussian Elimination to determine if the following system of equations has either **no solution, one solution, or infinite solutions**. If there is a single solution please write it explicitly, and if there are infinite solutions please **specify the full set of solutions**.

$$\begin{aligned} x+y+z &= 3 \\ 4x+3y+2z &= 5 \\ 7x+3y+4z &= 8 \end{aligned}$$

Solutions:
First we transcribe the linear system into a matrix form $\mathbf{A}\vec{x}=\vec{b}$.
From this point we can cast the problem into an augmented form $\left[\mathbf{A}|\vec{b}\right]$, which allows us to employ Gaussian elimination.

$$\mathbf{A}\vec{x}=\vec{b} \quad \longrightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 7 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \quad \longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 2 & 5 \\ 7 & 3 & 4 & 8 \end{array}\right]$$

We begin reducing our matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 2 & 5 \\ 7 & 3 & 4 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -7 \\ 0 & -4 & -3 & -13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -7 \\ 0 & -4 & -3 & -13 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 5 & 15 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

One can identify a unique solution $\vec{x}$ from this point, since the final row states $z=3$, then the second row states $y=7-2z=7-6=1$, and finally the top row says $x=3-y-z=3-1-3=-1$.

However, you can also continue row reducing back up to yield the identity on the left-hand side:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Since left side of our augmented form can be row-reduced to the identity, there is a unique solution. This solution is the right-hand side of the solved augmented form, $x=-1$, $y=1$, and $z=3$.

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \quad \square$$

(b) (4 points) **Find the null space $N(\mathbf{A})$ of the following matrix.**

$$\mathbf{A} = \begin{bmatrix} -6 & 8 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

Solutions:

Finding the null space of **A** means we need to find all vectors $\vec{x} \in \mathbb{R}^3$ that satisfy the expression $A\vec{x} = \vec{0}$. Performing a full row reduction on $\left[\mathbf{A}|\vec{0}\right]$ yields the following augmented form

$$\left[\begin{array}{ccc|c} -6 & 8 & 1 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_2 \to R_2 + \frac{1}{2}R_1} \left[\begin{array}{ccc|c} -6 & 8 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right] \xrightarrow[R_1 \to -\frac{1}{6}R_1]{R_1 \to R_1 - 8R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right].$$

Evidently we cannot reach row-echelon form due to the width of this matrix. So to find the null space for vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we must introduce at least one free parameter $z = \alpha$; this parameter choice is not unique, but it is the most obvious/direct choice. From this point we can use the rows in the reduced form to identify $x = -\frac{1}{2}\alpha$ and $y = -\frac{1}{2}\alpha$. Thus

$$N(A) = \text{span}\left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}. \quad \square$$

Notice that the span can be specified using any vector within the space (ie. you could have chosen a different α for the representation).

$$N(A) = \text{span}\left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

(c) (6 points) Ashwin has lost his op-amp! Let its location in 2D be denoted by the vector $\vec{x} \in \mathbb{R}^2$. **Set up a set of *linear* equations in the form $\mathbf{A}\vec{x} = \vec{b}$ and solve for $\vec{x}$** based on the provided information:

i. It is 2 units away from $(1,3)$.
ii. It is $\sqrt{10}$ units away from $(2,4)$.
iii. It is 3 units away from $(-2,1)$.

Solutions:
From the information, we construct the following system of equations.

$$(x-1)^2 + (y-3)^2 = 2^2 \implies x^2 - 2x + y^2 - 6y + 10 = 4 \implies \left(\frac{x^2+y^2}{2}\right) - x - 3y = -3 \quad (1)$$

$$(x-2)^2 + (y-4)^2 = 10 \implies x^2 - 4x + y^2 - 8y + 20 = 10 \implies \left(\frac{x^2+y^2}{2}\right) - 2x - 4y = -5 \quad (2)$$

$$(x+2)^2 + (y-1)^2 = 3^2 \implies x^2 + 4x + y^2 - 2y + 5 = 9 \implies \left(\frac{x^2+y^2}{2}\right) + 2x - y = +2 \quad (3)$$

Note that the position of the op-amp is located at $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.
Pick an equation to subtract out the x^2 and y^2 terms.
For example, using the second equation $(1)-(2)$ and $(3)-(2)$ would yield

$$\begin{aligned} x + y &= 2 \\ 4x + 3y &= 7, \end{aligned}$$

so one acceptable solution is

$$\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

From this stage we can row reduce $(R_2 \to R_2 - 4R_1)$ and scale $(R_2 = -1R_2)$ to achieve a sufficiently reduced matrix and identify $\vec{x}$:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad \square$$

Technically we could have instead subtracted (1) from the remaining equations, or subtracted (3) from the others. The only other equation we would gain is (3)-(1) which simplifies to $3x + 2y = 5$. All in all, the full array of equations that we can derive is presented with a 3×2 system below

$$\begin{bmatrix} 1 & 1 \\ 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}.$$

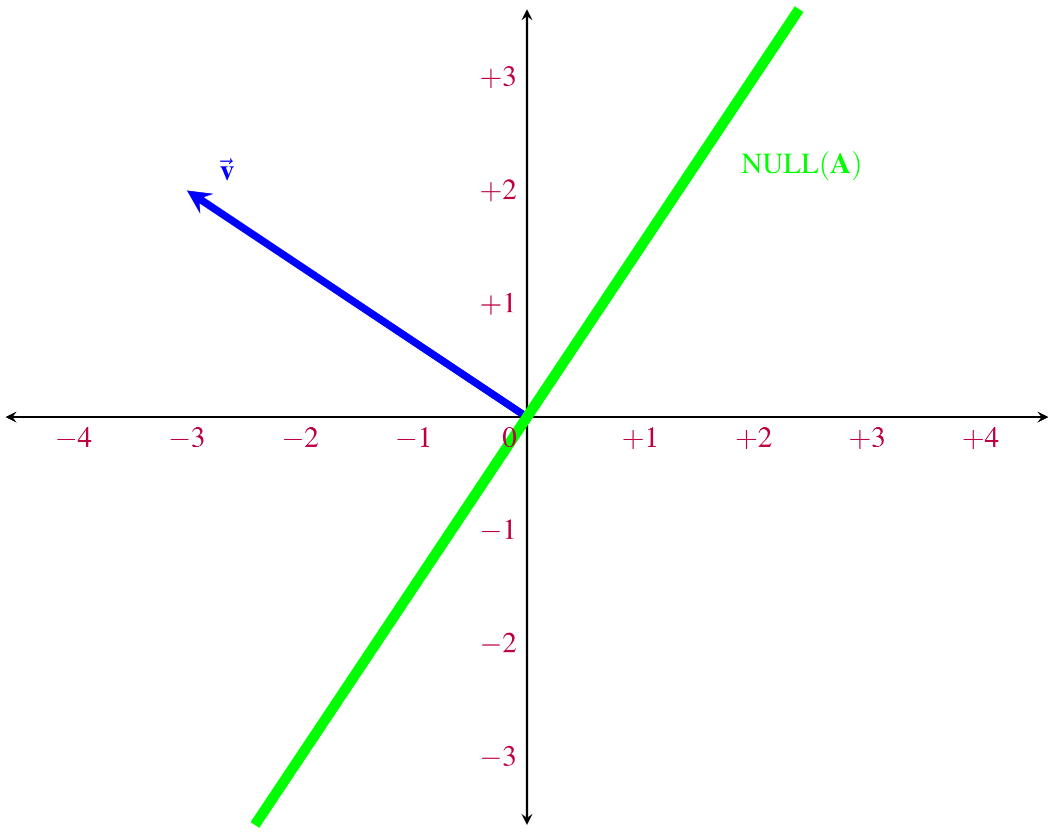
(d) (4 points) For this part you will need to sketch vectors on a 2D plane. Make sure your plot clearly labels the x and y axes.

i. **Plot the vector** $\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Label the vector and clearly indicate the vector components on the plot.

ii. On the same graph **plot the null space of the matrix** $\mathbf{A} = \begin{bmatrix} -3 & 2 \end{bmatrix}$.

Solutions:
We plot the vector $\vec{v}$ as an arrow. The null space $N(\mathbf{A})$ consists of all vectors $\vec{x} \in \mathbb{R}^2$ (width of $\mathbf{A}$) such that $-3x + 2y = 0$. We can solve this expression for a line on the plane $y = \frac{3}{2}x$, which describes the null space for $\vec{x}$.

A quicker method for identifying the null-space is to recognize that $\vec{x}$ has to be orthogonal to the row of $\mathbf{A}$, which is equal to our prior vector $\vec{v}$. Thus the null space is the line perpendicular to $\vec{v}$. The plot should reflect this.



(e) (4 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. **Compute the following inner product**:

$$\left\langle 2\vec{x} + \vec{y}, \frac{1}{2}\vec{y} \right\rangle$$

Solutions:

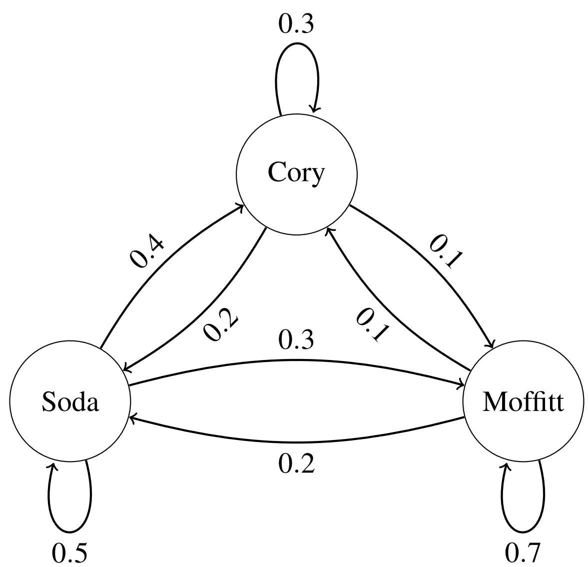
Using inner product properties decomposes this expression into

$$\begin{aligned}
\left\langle 2\vec{x} + \vec{y}, \frac{1}{2}\vec{y} \right\rangle &= \left\langle 2\vec{x}, \frac{1}{2}\vec{y} \right\rangle + \left\langle \vec{y}, \frac{1}{2}\vec{y} \right\rangle \\
&= 2 \cdot \frac{1}{2}\langle \vec{x}, \vec{y} \rangle + \frac{1}{2}\langle \vec{y}, \vec{y} \rangle \\
&= (1)(3) + (2)(4) + \frac{1}{2}\left(3^2 + 4^2\right) \\
&= 23.5
\end{aligned}$$

2. (16 points) Transitioning Back to Campus

UC Berkeley administrators are drafting up the social distancing guidelines for reintroducing engineering students back onto campus. They know these students typically spend a lot of time in Soda Hall, Cory Hall, and Moffitt Library and they need to determine how many students can be re-introduced without violating any Covid-related building occupant capacities. They have an initial transition-state model acquired from prior years, but need your help.

(a) (6 points) Prior on-campus student traffic data lead administrators to assemble the following transition diagram describing how students move between these buildings (where each arrow represents the proportion of students moving).



The current number of students in each building at time-step t is given by the state vector $\vec{x}[t]$ defined as:

$$\vec{x}[t] = \begin{bmatrix} x_C[t] \\ x_S[t] \\ x_M[t] \end{bmatrix} = \begin{bmatrix} \text{number of students in Cory at time } t \\ \text{number of students in Soda at time } t \\ \text{number of students in Moffitt at time } t \end{bmatrix}$$

i. **Explicitly write out the transition matrix T from the provided diagram** such that $\mathbf{T}\,\vec{x}[t] = \vec{x}[t+1]$.

ii. Does this model account for all students leaving or staying at each of these three buildings? In other words, **is the system conservative?** Justify your answer.

Solutions:

Part i.

$$T = \begin{bmatrix} C \to C & S \to C & M \to C \\ C \to S & S \to S & M \to S \\ C \to M & S \to M & M \to M \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}$$

Part ii.

No, not all students have been accounted for.

In general, the total number of students will change after applying the transition matrix since the transition matrix is *not* conservative. This is because not all columns of T sum to 1; the sums are 0.6, 1.2, and 1.0.

(b) (10 points) Berkeley administrators have just modified the original transition matrix model based on certain courses/labs remaining in remote operation, hence the new transition matrix

$$\mathbf{M} = \begin{bmatrix} 0.6 & 0.0 & 0.0 \\ 0.2 & 0.4 & 0.6 \\ 0.2 & 0.6 & 0.4 \end{bmatrix}.$$

State guidelines impose limits on the typical number of students occupying each building. To predict if the number of students in each building will meet guidelines, you decide to examine the steady-state behavior of the transition system.

i. You are given that: $\lambda_1 = 1$, $\lambda_2 = -0.2$, and $\lambda_3 = 0.6$.
Identify a steady state vector $\vec{x}_{steady}$ such that $\mathbf{M}\vec{x}_{steady} = \vec{x}_{steady}$.

ii. State guidelines impose the following limits on the number of students occupying Cory, Soda, and Moffitt:

$$\vec{x}_{limit} = \begin{bmatrix} 100 \\ 60 \\ 80 \end{bmatrix}.$$

It is also anticipated that the following number of students will be in each building at the start of the day $\vec{x}_0 = [20, 50, 70]^T$. **Argue whether or not the state guidelines $\vec{x}_{limit}$ will be satisfied in the steady state (after an infinite number of time-steps occur).**

Solutions:

Part i.
The steady state vector $\vec{x}_{steady}$ is the eigenvector corresponding to the eigenvalue of 1. To solve for the eigenvector that satisfies $\mathbf{M}\vec{v} = 1\vec{v}$ we must identify the null space of $(\mathbf{M} - 1\ \mathbf{I})$, since we know $(\mathbf{M} - 1\ \mathbf{I})\vec{v} = \vec{0}$.

$$\left[\mathbf{M} - 1\ \mathbf{I} \mid \vec{0}\right] = \left[\begin{array}{ccc|c} -0.4 & 0.0 & 0.0 & 0 \\ 0.2 & -0.6 & 0.6 & 0 \\ 0.2 & 0.6 & -0.6 & 0 \end{array}\right] \xrightarrow[R_3 \to -R_3 - (1/2)R_1]{R_2 \to R_2 + (1/2)R_1} \left[\begin{array}{ccc|c} -0.4 & 0 & 0 & 0 \\ 0 & -0.6 & 0.6 & 0 \\ 0 & -0.6 & 0.6 & 0 \end{array}\right]$$

Evidently the final two rows cancel, leaving us with an underdetermined system. We set $z = \alpha$ to a free parameter and then conclude $y = z = \alpha$ from row 2 and that $x = 0$. This yields a 1-dimensional eigenspace with a single eigenvector:

$$\vec{x}_{steady} = \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ for any } \alpha \in \mathbb{R}.$$

The final steady state must be of the form $\vec{x}_{steady}$ for some $\alpha \in \mathbb{R}$. □

Part ii.
Since the matrix T is conservative, we know that the total number of students in the steady state must match the total number of students at time 0. The total number of students in $\vec{x}_0$ is $20 + 50 + 70 = 140$. This must equal the total number of students in steady state: $\alpha * 0 + \alpha * \frac{\sqrt{2}}{2} + \alpha * \frac{\sqrt{2}}{2} = \alpha\sqrt{2}$. Thus

$\alpha\sqrt{2} = 140$, $\alpha = \frac{140}{\sqrt{2}}$ and the steady state will be $\vec{x}_{steady} = \begin{bmatrix} 0 \\ 70 \\ 70 \end{bmatrix}$. The 70 students in Soda will exceed the 60 student limit imposed by the state guidelines.

3. (16 points) Negative Resistance Circuit

While waiting for lab checkoff you decide to fiddle with some op-amp topologies, and stumble upon the circuit below (Figure 1) that behaves like a negative-valued resistor! In this question you will be guided through a method for finding the equivalent resistance of the circuit. Afterwards we will investigate one potential application of this circuit.

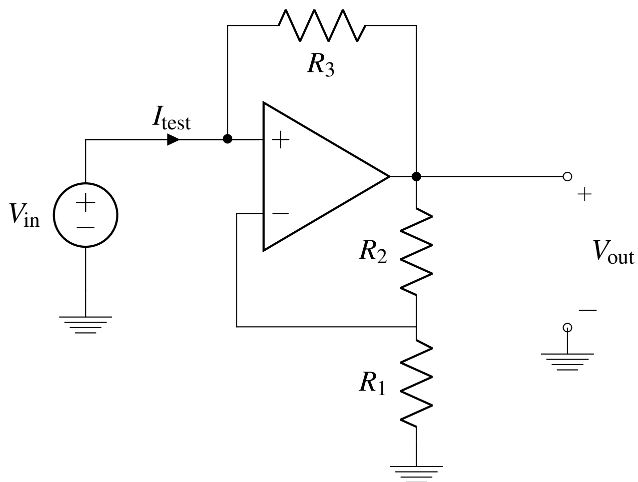


Figure 1: Op-amp circuit that behaves like a negative-valued resistor.
I_{test} is defined as the current from the voltage source V_{in} towards the u_+ input op-amp node.

(a) (6 points)

i. **Find an equation for V_{out} in terms of V_{in}, I_{test}, and R_3.**

ii. **Find an equation for V_{out} in terms of V_{in}, R_1, and R_2.**

iii. The equivalent resistance looking into the circuit will be the voltage at the node divided by the current going into the node: $R_{\text{eq}} = \frac{V_{\text{in}}}{I_{\text{test}}}$. **What is the equivalent resistance of this circuit?** You should use the results from parts (i) and (ii), and your answer should not contain V_{out}.

Solutions:

i. Since the only element between the V_{in} and V_{out} nodes would be R_3, and since I_{test} is already specified, we can immediately find the relation between these using Ohm's law $V_3 = I_{test}R_3$. This yields

$$V_{\text{out}} = V_{\text{in}} - I_{\text{test}}R_3. \quad \square$$

ii. For this part we must first acknowledge that the op-amp circuit forms a negative feedback system, which admits use of the golden rules to say $u_- = u_+ = V_{in}$. Next we need to determine the current through R_1, which can be given by Ohm's law $I_0 = \frac{V_{in}-0}{R_1}$. Note in our convention (numerator of the last equation) that I_0 flows towards ground.

Since another golden rule states that no current can flow in/out of op-amp input terminal, it must be by KCL that I_0 also flows through R_2 (and is bound for R_1). This lets us work back Ohm's law on R_2 to identify V_{out} in terms of known quantities $I_0 = \frac{V_{out}-u_-}{R_2}$. All of these findings lead us to the final equation

$$V_{\text{out}} = \left(1 + \frac{R_2}{R_1}\right)V_{\text{in}}. \quad \square$$

iii. For this circuit to function as expected, the equations from (i) and (ii) must be consistent with each other. So to identify I_{test} we equate the expressions found in (i) and (ii) and solve for the current. Then substitute this back in to solve for R_{eq}.

$$\begin{aligned} V_{\text{in}} - I_{\text{test}}R_3 &= \left(1 + \tfrac{R_2}{R_1}\right)V_{\text{in}} \\ I_{\text{test}} &= V_{\text{in}}\left(-\tfrac{R_2}{R_1R_3}\right) \\ R_{\text{eq}} &= \tfrac{V_{\text{in}}}{I_{\text{test}}} = -\tfrac{R_1R_3}{R_2}. \quad \square \end{aligned}$$

(b) (4 points) In lab you are using a current source to test a load resistance, but you find that the load current I_L depends on the load resistance R_L. You infer that the current source has an internal source resistance R_0 and come up with the following model (Figure 2) for your circuit.

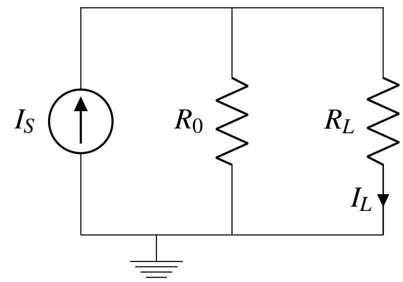


Figure 2: Model diagram for a current source with internal resistance R_0.

What is I_L in terms of given variables?
Your solution should be in terms of R_0, R_L, and I_S.

Solutions:
Set the node voltage shared by the top of R_0 and R_L as u. We know from our current conservation law that $I_S = I_0 + I_L$, where I_0 passes through R_0 and is bound for ground. Further we know $I_0 R_0 = I_L R_L = u$. The combination of these two equations yield the result $I_S = \frac{u}{R_0} + \frac{u}{R_L} \rightarrow u = \frac{I_S}{R_0 + R_L}$. Finally we can work Ohm's law back to identify $I_L = \frac{u}{R_L}$ to produce

$$I_L = \left(\frac{R_0}{R_L + R_0}\right) I_S. \qquad \square$$

Alternatively, one could start from the equivalent resistance $R_{EQ} = \frac{R_0 R_L}{R_0 + R_L}$ of Figure 2 to identify u.

(c) (6 points) You decide to use the op-amp circuit from part (a) in order to make the load current in part (b) independent of R_L. The circuit from part (a) can be modeled as a resistor with resistance $R_a < 0$ (a negative value). This new circuit element R_a will be wired in parallel with the current source as shown in Figure 3.

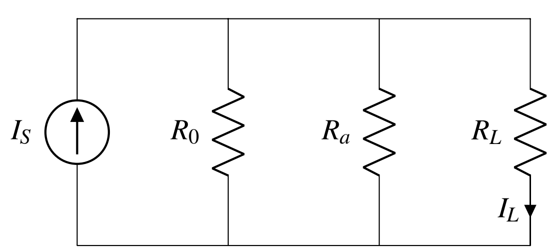


Figure 3: Diagram for the application of the *negative resistance* circuit to a current source.

Choose a value for R_a such that the current through the load resistors I_L is equal to the current through the source current I_S. Show that $I_L = I_S$ with your chosen R_a (i.e. do not just guess a value).

Solutions:
In an ideal scenario, the internal resistance R_0 would be infinite so that all current from I_S flows exclusively through the load R_L. With the inclusion of $R_a < 0$ in parallel with R_0, we may just be able to produce an effectively-infinite equivalent resistance.

The trick here is to examine the mathematical equation for the equivalent resistance of R_0 and R_a. You may notice with the specific choice of $R_a = -R_0$ that the denominator in our R_{eq} expression goes to zero, thus producing an effectively-infinite resistance just like we had hoped for!

$$R_{eq} = \frac{R_0 R_a}{R_0 + R_a} = \frac{R_0(-R_0)}{R_0 + (-R_0)} \rightarrow \infty$$

With this choice of R_a implemented, the combined equivalent resistance of R_a and R_0 behave likes an open wire and we can thus redraw our circuit as:

Applying KCL to this re-drawn diagram yields $I_S = I_L$. □

4. (18 points) Designing a light meter

Our plants keep dying from not getting enough sun! To prevent this we want to design a circuit to measure the light the plant gets. We will start with a photodetector, which we can model as a current source I_s. When the plants are getting sufficient sun exposure, the current source outputs 5 nA $= 5 \times 10^{-9}$A. Conversely, when they are *not* getting enough sun exposure the current source outputs 0 nA.

(a) (6 points) We wire up the current source I_s into the capacitor circuit shown in Figure 4 below.

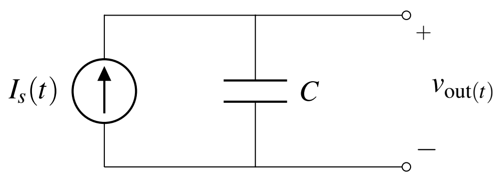


Figure 4: Light meter circuit, where the current source $I_s(t)$ models the photodetector.

Find an expression for $v_{out}(t)$ in terms of $I_s(t)$, C, and t when under *constant* light exposure ($I_s(t) = 5$ nA). Then identify the capacitor value C such that, after 1 hour under exposure, the capacitor voltage is $V_{out} = 5V$. Assume the initial voltage on the capacitor is 0V.

Solutions:
The charge on a capacitor is defined through the equation $C = \frac{Q}{V}$, and so the current being applied ($I = dQ/dt$) to a capacitor follows by taking the time-derivative of the capacitor equation $I = \frac{dQ}{dt} = C\frac{dV}{dt}$. This can be rearranged to find V in terms of the current source and the capacitor value:

$$\frac{dV_{out}}{dt} = \frac{1}{C}I_s(t) \longrightarrow \int_0^t \frac{dV_{out}(t')}{dt'}dt' = \frac{1}{C}\int_0^t I_s(t')dt' \longrightarrow V_{out}(t) = \frac{1}{C}\int_0^t I_s(t')dt' + V_{out}(0).$$

Since there is no initial voltage across the capacitor at $t = 0$, we can simplify our final solution to

$$V_{out}(t) = \frac{1}{C}\int_0^t I_s(t')\,dt'.$$

Since we are looking for $V_{out}(t)$ when under constant sun exposure, we can substitute the constant value $I_s(t) = 5$nA into the integral. This yields

$$V_{out}(t) = \left(\frac{(5 \times 10^{-9}\text{ A})}{C}\right)t. \quad \square$$

To determine the capacitance C needed to meet our voltage condition, we first recognize that the voltage expression simplifies thanks to the photodetector behaving like a constant current source

$$V_{out}(t) = \left(\frac{1}{C}\right) I_s\, t \longrightarrow C = \frac{(5 \times 10^{-9}\, A)(60 \times 60\, s)}{5\, V} = 3600 \times 10^{-9}\left(\frac{A\, s}{V}\right).$$

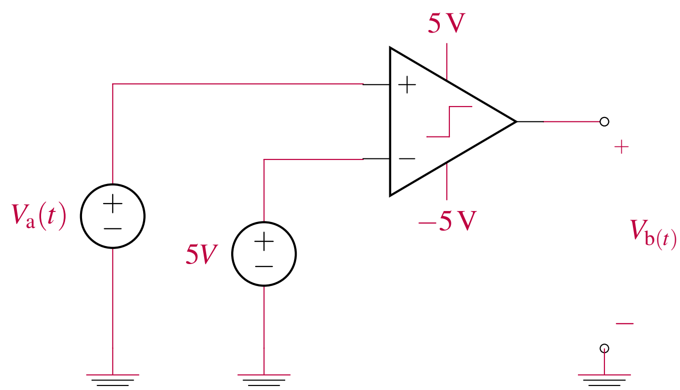
Thus $C = 3.6\mu$F. $\square$

(b) (6 points) We would like to use the previous circuit to power a separate LED device that indicates the state of sun exposure on our plant. The device indicates sufficient sun exposure when +5V is applied across it, and conversely indicates the plant is critically underexposed when -5V is applied across it. **Design a circuit *using a comparator* that outputs +5V once the plant has received at least 1 hour of full exposure, and otherwise outputs -5V.** You have a voltage source $V_a(t)$ which corresponds to $V_{out}(t)$ from part (a).

Regardless of your answer in part (a) assume $V_a(t)$ functions exactly as previously described; so $V_a(t) = 5V$ after an hour of full exposure. You may use as many voltage sources as you would like. Label the comparator rails and any voltage sources you include (with explicit voltage values).

Solutions:

We see from part (a) that $V_a(t)$ is directly proportional to the exposure time. Therefore $V_a(t)$ will output 5V after the sensor has been exposed for 1 hour and less than 5V otherwise. By putting a voltage of 5V in the negative terminal of the op-amp, -5V will output until 1 hour is reached.



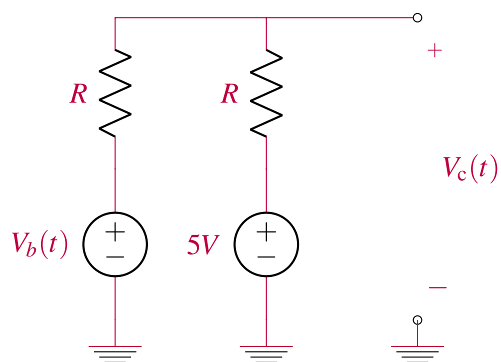
(c) (6 points) We want to use the comparator output from the previous circuit to talk to a microcontroller. However, your microcontroller can only read 0V to 5V, instead of the -5V to 5V output voltage from the comparator in part (b). Furthermore, you do not have access to any other comparator. **Design a circuit, without using a comparator, that scales and shifts an input voltage in the range** $-5V \leq V_{in} \leq +5V$ **to produce an output voltage** $0V \leq V_{out} \leq +5V$**.** Use the voltage source $V_{in} = V_b(t)$ to model the output of part (b). **You are limited to only use circuit elements provided with your lab**, which entails 4 resistors, two op-amps, and one constant voltage source. For any resistors or voltage sources that you use in your design, you may pick any component value, but please clearly label and specify its value.

Hint: There are multiple possible solutions to this sub-part.

Solutions:
Ultimately we would like to assemble a circuit which performs the following voltage map:

$$V_c(t) \;=\; \frac{1}{2}V_b(t) \;+\; \frac{5}{2}V.$$

One quick option is to implement a voltage summer circuit:

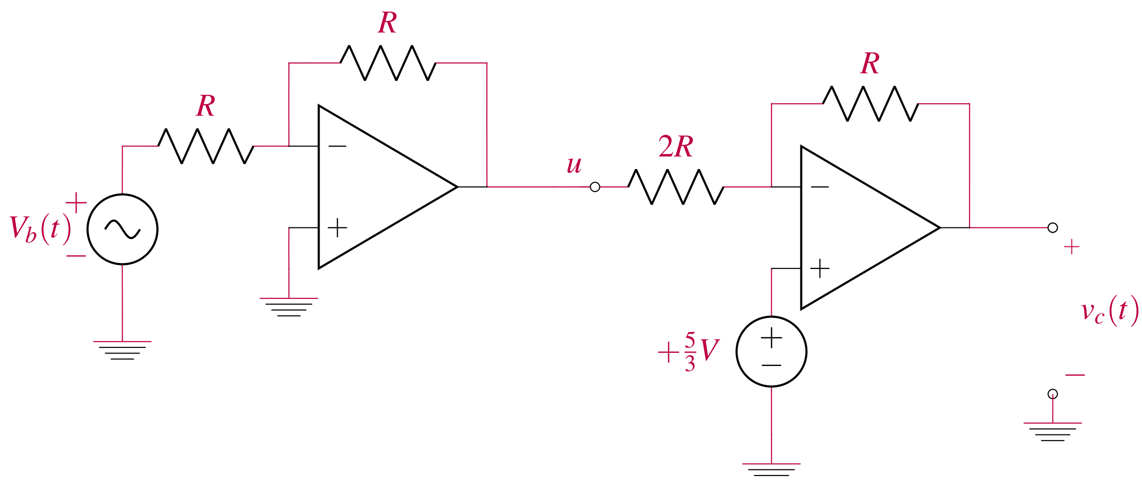


By setting the two resistors equal in value (they can be any resistance), we produce a circuit which outputs the desired map (which follows from a quick super-position assessment)

$$V_c(t) \;=\; \left(\frac{1}{2}\right)V_b(t) \;+\; \left(\frac{1}{2}\right)5\;V.$$

Notice that $V_b = +5$V produces $V_c = \left(\frac{5}{2}+\frac{5}{2}\right)\;V \;=\; +5\;V$, and $V_b = -5$V produces $V_c = \left(\frac{-5}{2}+\frac{5}{2}\right)\;V \;=\; 0\;V$, just as desired. □

Alternatively, one could implement 2 inverting amplifiers with an additional voltage source to accomplish this task. First you must invert the input voltage $V_b(t) \;\rightarrow\; -V_b(t)$, then you must scale and shift the output using another inverting amplifier.



The first stage simply inverts $V_b(t)$ since the output at node u is

$$u \;=\; 0 - R\left(\frac{V_b - 0}{R}\right) = -\frac{R}{R}V_b(t) = -V_b(t).$$

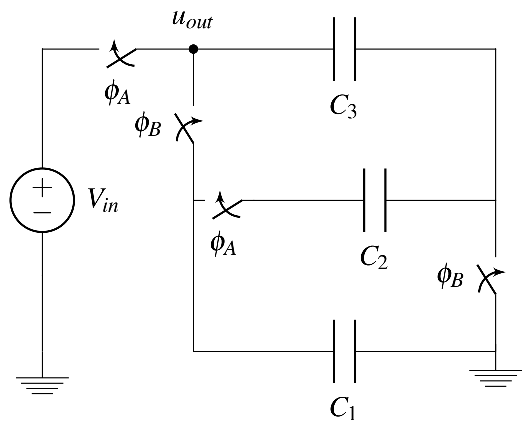
Now the output from the second stage produces

$$V_c(t) \;=\; \left(\frac{5}{3}V\right) - \frac{R}{2R}\left(u - \frac{5}{3}V\right) \;=\; \left(\frac{5}{3}V + \frac{5}{6}V\right) + \frac{1}{2}V_b(t) \;=\; \frac{1}{2}V_b(t) + 2.5\,V.$$

Thus our circuit reproduces the correct mapping of the $V_b(t)$ input voltage. $\square$

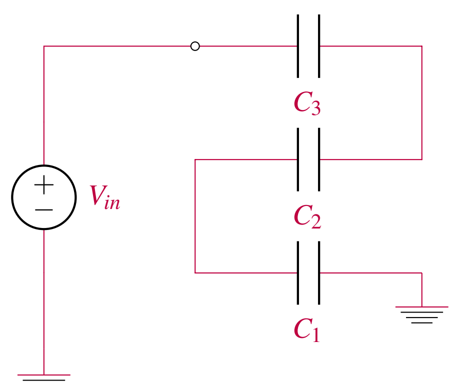
5. (16 points) Charge sharing check-in

(a) (8 points) Let us analyze the capacitor circuit shown below. Let us set all of the capacitors to have the same capacitance $C_1 = C_2 = C_3 = C$. Assume that all capacitors start without any initial charge, i.e. they are completely discharged before phase A.



The switches for phase A close first and the capacitors charge up completely. Those switches are then disconnected and the switches for phase B are closed. **What is the voltage at node u_{out} in phase B in terms of C and V_{in}?** Draw out the two phases of the circuit for partial credit.

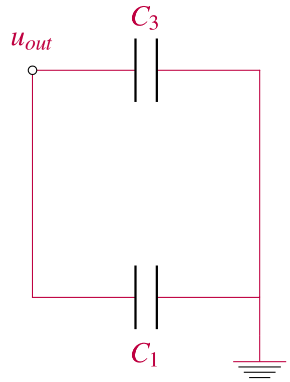
Solutions:
First we draw out the circuit in phase A:



We must find the equivalent capacitance for the capacitors in series: $C_{EQ} = C/3$.

With this simplified circuit, we identify a charge of $Q_A = +\frac{1}{3}CV_{in}$ on each positive plates (and conversely $-CV_{in}/3$ on the negative plates).

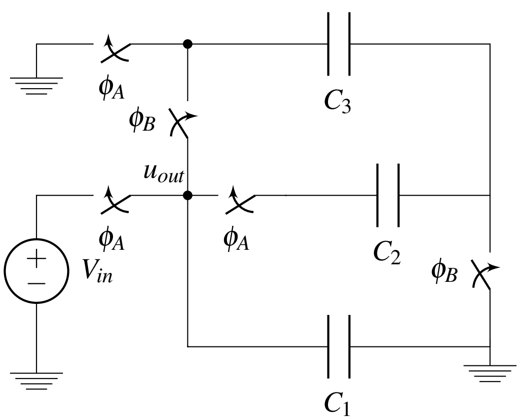
Next, we draw the circuit in phase B:



At the floating node u_{out}, we see the positive plates of the capacitors C_1 and C_3. The charge stored on those plates from phase A is $CV_{in}/3 + CV_{in}/3 = 2CV_{in}/3$. The charge at u_{out} in phase B is $2Cu_{out}$.

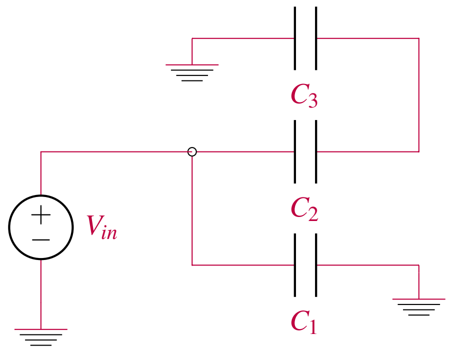
Setting the charges equal, we find that $u_{out} = V_{in}/3$.

(b) (8 points) In this part we will analyze a slightly different circuit, shown below. Let us again set all of the capacitors to have the same capacitance $C_1 = C_2 = C_3 = C$. Assume that all capacitors start without any initial charge, i.e. they are completely discharged before phase A. The switches for phase A close first and the capacitors charge up completely. Those switches are then disconnected and the switches for phase B are closed.



What is the voltage at u_{out} in phase B in terms of C and V_{in}? Draw out the two phases of the circuit for partial credit.

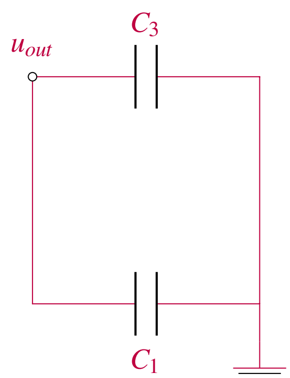
Solutions:
First we draw the circuit in phase A:



We first find the equivalent capacitance of the capacitors C_2 and C_3 in series on the top, which is $C/2$.

Thus the charge stored on the top two capacitors C_2 and C_3 in phase A is $CV_{in}/2$, and the charge stored on the bottom capacitor C_1 is CV_{in}.

Next we draw the circuit in phase B.



At the floating node u_{out}, we see the positive plate of the capacitors C_1 and the *negative* plate of C_3. The charge stored on those plates from phase A is $CV_{in} - CV_{in}/2 = CV_{in}/2$. The charge at u_{out} in phase B is $2Cu_{out}$.
By setting these equal we find that $u_{out} = V_{in}/4$.

6. (16 points) Transatlantic Telegraph Cable

The year is 1956, and secret agents Alice & Bob have been deployed to New York and London respectively. Alice regularly needs to send Bob sensitive information, and they just caught word of a recently established transatlantic telegraph cable TAT-1 between the continents, which has a much faster communication speed than mailing letters.

(a) (4 points) Alice and Bob use conventional mail to agree on a specific binary code for conveying their secret messages. Alice found a 6-element code in one of her old training manuals, but unfortunately one of the numbers in code is illegible:

$$\vec{s}[n] = [\,+1\,,\,-1\,,\,+1\,,\,+1\,,\,-1\,,\,\gamma\,]$$

(where γ represents the missing entry, which will be either +1 or -1). Luckily her manual also provides a diagram of this code's auto-correlation function, plotted in Figure 5 below.

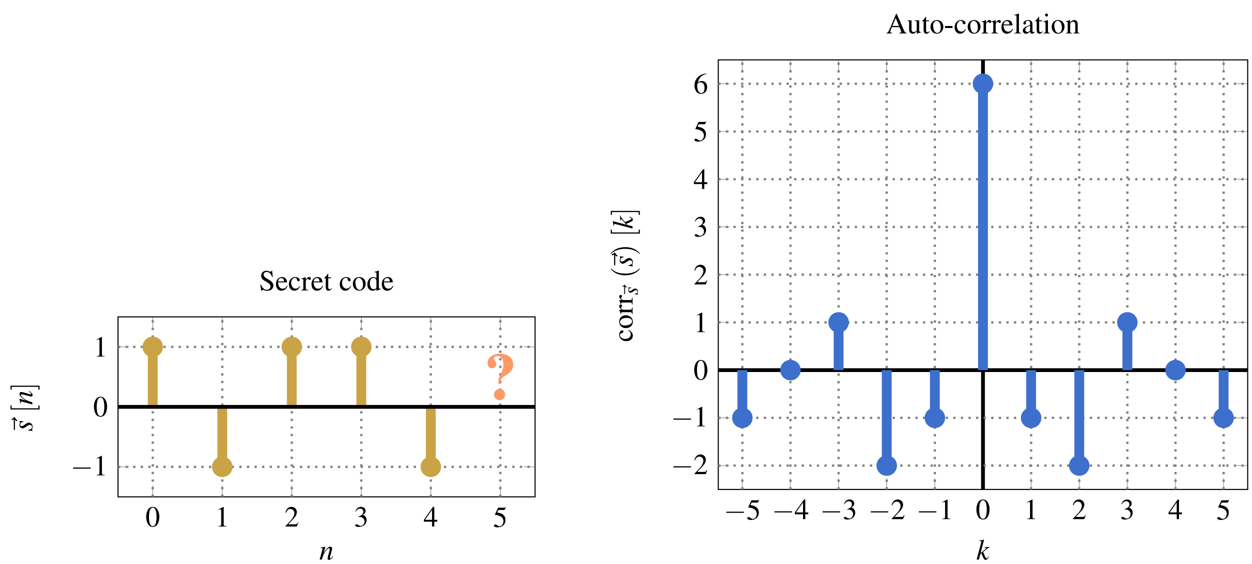


Figure 5: Alice's secret code (left) has a length of 6 elements but the last entry $\vec{s}[5] = \gamma$ is unknown. The auto-correlation (right) $\mathrm{corr}_{\vec{s}}(\vec{s})[k]$ of the secret code.

Identify the missing entry γ (either $+1$ or -1) of the code from Alice's manual. Provide explicit reasoning to justify your answer.

Solutions:
Recall the definition for auto-correlation:

$$\mathrm{corr}_{\vec{s}}(\vec{s})[k] \;=\; \sum_{n=-\infty}^{+\infty} s[n]\; s[n-k].$$

Since there are only 2 possible values for γ, let's first try $\gamma = +1$ and then compute the auto-correlation of $\vec{s}$. This yields (for $-5 \le k \le 5$)

$$\mathrm{corr}_{\vec{s}}(\vec{s})[k] = [\,1\,,\,-2\,,\,3\,,\,0\,,\,-3\,,\,6\,,\,-3\,,\,0\,,\,3\,,\,-2\,,\,1\,].$$

Unfortunately this doesn't match the given auto-correlation sequence.

Alternatively we may verify $\gamma = -1$ by computing the auto-correlation with this entry (for $-5 \leq k \leq 5$)

$$\text{corr}_{\vec{s}}(\vec{s})[k] = [\ -1\ ,\ 0\ ,\ 1\ ,\ -2\ ,\ -1\ ,\ 6\ ,\ -1\ ,\ -2\ ,\ 1\ ,\ 0\ ,\ -1\ ].$$

This matches the given auto-correlation sequence. Therefore the missing number is $\gamma = -1$. □

Technically one doesn't need to check the entire auto-correlation sequence. Full credit can be awarded for computing the auto-correlation at a specific non-zero k value ($k \neq 0$) for $\gamma = \pm 1$ and compare with the corresponding value in the plot to reach the correct conclusion that the missing number is $\gamma = -1$.

(b) (4 points) Unfortunately, enemy counter-intelligence managed to intercept Alice's mail to Bob, so she decides to use a new secret code (shown in Fig. 6):

$$\vec{s}[n] = [\,1\,,\ -1\,,\ 1\,,\ -1\,,\ -1\,,\ -1]$$

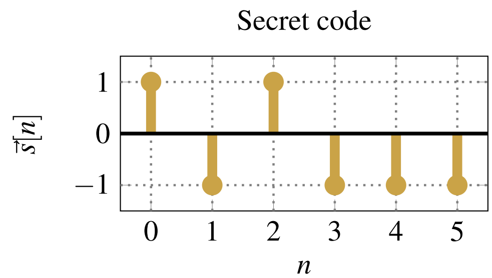


Figure 6: Alice's newest secret code (top).

In order to communicate through the transatlantic cable, Alice and Bob need to determine the transmission delay between them. This will help Bob find the starting time stamp of Alice's messages. Alice starts transmitting the secret code repeatedly. She transmits the first element of each 6-element code at time stamps $n = [-12, -6, 0, +6, +12]$. At the receiving side, Bob sees the code but with some delay. Bob's best guesses of the delay are $3T$, $4T$, or $5T$, where T denotes the time interval between adjacent time stamps.

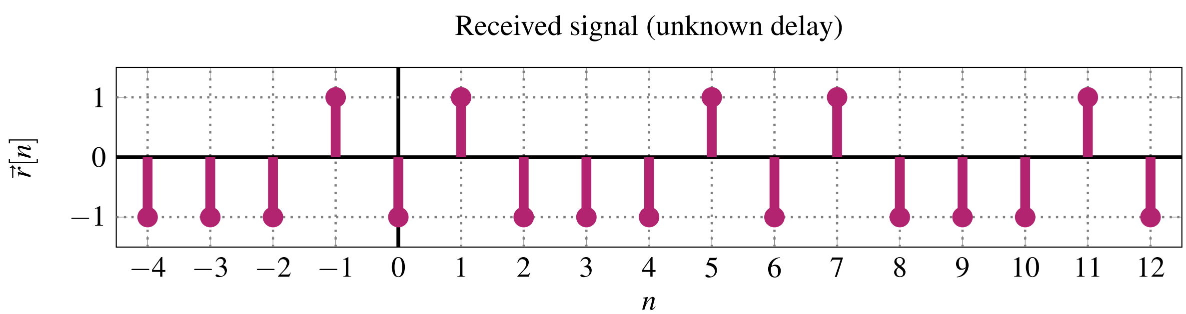


Figure 7: Bob's received signal, shifted by some unknown delay (bottom)

Bob receives the signal $\vec{r}[n]$ shown in the lower Figure 7 above. **Based on Bob's guesses, what is the actual time delay in terms of T? Justify mathematically.** You can assume that the delay is an integer number of T (i.e. no decimals).

Solutions: Compute the cross-correlation $\text{corr}_{\vec{r}}(\vec{s})[k]$ for $k = 3, 4, 5$, we get the following results:

$$\text{corr}_{\vec{r}}(\vec{s})[k=3] = 2$$
$$\text{corr}_{\vec{r}}(\vec{s})[k=4] = -2$$
$$\text{corr}_{\vec{r}}(\vec{s})[k=5] = 6$$

A shift of $k = 5$ matches the secret code $\vec{s}[n]$ with the received signal $\vec{r}[n]$, therefore the actual time delay is $5T$.

(c) (4 points) Now that Alice and Bob have determined the delay, Alice can send messages using the secret code. Alice's messages contain binary symbols (composed ± 1 values), and she encodes her messages by multiplying each of her message symbols with the 6-element secret code. So if she sends a message with 18 elements, it must contain $18/6 = 3$ symbols of information. For this part assume there is no time delay between transmission and reception.

Alice sends a 18-element-long signal to Bob. However, some noise corrupts the signal, so the signal Bob receives $\vec{r}[n]$ contains imperfect samples instead of +1 and -1. He decides to take the cross correlation with the secret code to try to decode the message.

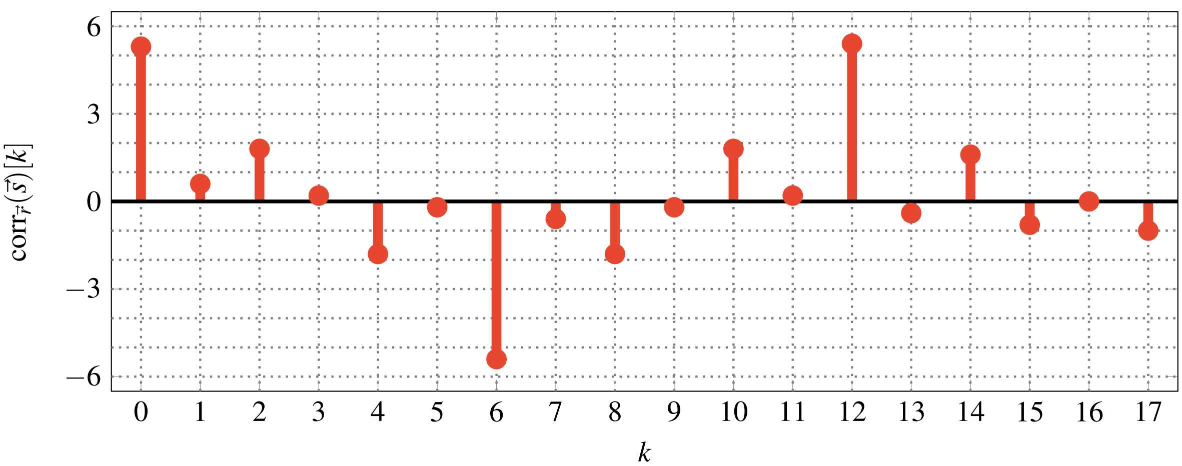


Figure 8: Cross correlation of the secret code with Bob's received signal, containing 3 symbols of information. $k < 0$ and $k > 17$ are ignored.

The cross correlation with the secret code for shifts $k = 0$ through $k = 17$ is:

$\text{corr}_{\vec{r}}(\vec{s})[k] = [5.3, 0.6, 1.8, 0.2, -1.8, -0.2, -5.4, -0.6, -1.8, -0.2, 1.8, 0.2, 5.4, -0.4, 1.6, -0.8, 0, -1]$

Based on the cross-correlation with the received waveform, extract the message sent to Bob. Justify your answer.

Solutions:

Recall the definition for cross-correlation

$$\text{corr}_{\vec{r}}(\vec{s})[k] = \sum_{n=-\infty}^{+\infty} r[n]\, s[n-k].$$

We know that each 6 transmitted elements corresponds to one message symbol. Since we assume no delay, we can look at the cross correlation with shifts $k = 0, 6, 12$.

$$\begin{aligned}
\text{corr}_{\vec{r}}(\vec{s})[k=0] &= +5.3 \\
\text{corr}_{\vec{r}}(\vec{s})[k=6] &= -5.4 \\
\text{corr}_{\vec{r}}(\vec{s})[k=12] &= +5.4
\end{aligned}$$

The first 6 elements have a positive correlation with the secret code, so the first symbol is +1. The second 6 elements have a negative correlation, meaning they are more similar to the secret code multiplied by -1. This means the second symbol is -1. Finally, the third symbol is +1 for a similar reason as the first symbol.

Therefore, Alice's message is $[\, +1\ ,\ -1\ ,\ +1\,]$. □

(d) (4 points) Enemy forces have cut the cable to prevent Alice & Bob from communicating. Alice can go underwater to fix the break, but she must first identify the break location in the Atlantic ocean. She expects the broken point to echo her signal back along the cable, i.e. if she transmits her signal, it will reflect at the break and she will receive the signal back after some delay. She decides to transmit her 6-bit code

$$\vec{s}[n] = [\ 1\ ,\ -1\ ,\ 1\ ,\ -1\ ,\ -1\ ,\ -1\ ]$$

and monitor the signal $\vec{r}[n]$ that echoes back. She computes the cross-correlation of the echo with the secret code, i.e. $\text{corr}_{\vec{r}}(\vec{s})[k]$, shown in Figure 9.

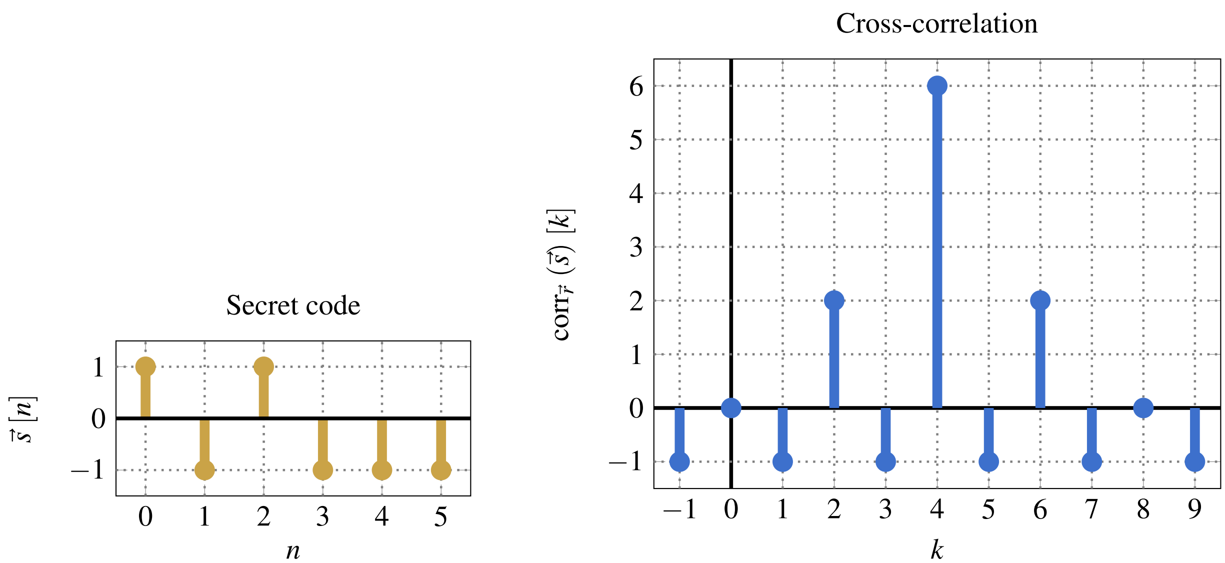


Figure 9: 6-bit secret code (left).
Cross-correlation (right) $\text{corr}_{\vec{r}}(\vec{s})[k]$ of the echo with the secret code.

Assume the time interval between adjacent time stamps is $T = 1$ ms $= 10^{-3}$ s, and that the signal travels in the telegraph cable at a speed of $v = 2 \times 10^8$ m/s (the speed of light is $c \approx 3 \times 10^8$ m/s in a vacuum, but in the cable medium it is $v = \frac{2}{3}c$). **How far is the broken point from Alice's location?**

HINT: Remember to account for the fact that the received signal has taken a full round trip, which is double the distance from Alice to the break.

Solutions:
The peak of the cross-correlation occurs at $k = 4$, so we know the time delay between the transmitted signal and the echo is $4T = 4$ ms. Note that the signal travels from Alice's location to the broken point then back to Alice's location, and the round-trip time delay is $2\times$ the single-trip time delay. Therefore, the distance from Alice's location to the broken point is

$$\frac{1}{2}(4 \text{ ms} \times 2 \times 10^8 \text{ m/s}) \;=\; 4 \times 10^5 \text{ m} \;=\; 400 \text{ km}.$$

7. (16 points) Cool Predictions

You have just been contracted by PG&E to predict daily energy use by the UC Berkeley campus based on local weather conditions. They have provided data of last year's daily energy usage along with corresponding weather reports.

(a) (5 points) You hypothesize a linear model based on the phase of the moon (represented as an integer between 0 and 8) and the season (represented as an integer between 0 and 3).

$$E = \alpha_P x_P + \alpha_S x_S \tag{4}$$

where E is the daily energy usage (in kilowatt-hours), x_P represents the moon phase, and x_S corresponds to the seasons. To get an initial approximation for the model parameters α_P and α_S, you sample data from 3 days to set up a linear system; the data have been printed in Table 1 below.

$\mathbf{x_P}$	$\mathbf{x_S}$	**E**
1	0	4
0	2	2
1	1	5

Table 1: Campus daily energy-usage
(x_P is the moon phase, x_S is the season, and E is the energy usage)

Explicitly set up the linear system of equations $\mathbf{D}\vec{a} = \vec{E}$, then from this system compute the least-squares solutions for $\hat{\vec{a}} = \begin{bmatrix} \alpha_P \\ \alpha_S \end{bmatrix}$.

Solutions:
We must first write out the data into a linear system:

$$\mathbf{D}\vec{a} = \vec{E} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_P \\ \alpha_S \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

Unfortunately the columns of $\mathbf{D}$ are not orthogonal, so there is no short-cut to determining the least squares solution. Next we employ the least squares formula to identify $\hat{\vec{a}} = (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\vec{E}$.

$$\begin{aligned} \hat{\vec{a}} = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \right)^{-1} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \\ = \left(\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \right)^{-1} & \begin{bmatrix} 9 \\ 9 \end{bmatrix} \\ = \frac{1}{9} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 9 \\ 9 \end{bmatrix} \end{aligned}$$

$$\hat{\vec{a}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \square$$

(b) (4 points) After speaking with the on-campus facility experts, you realize that a majority of energy is actually used for heating/air conditioning for indoor facilities (affected by outdoor temperature) and the outdoor pools (affected by wind conditions). This guides you to a new prediction model

$$E = \alpha_T (x_T - 15)^2 + \alpha_W \, x_W + \alpha_R \tag{5}$$

where E is daily energy usage (in kilowatt-hours), x_T corresponds to temperature (in Celsius), x_W corresponds to wind speed (in meters per second), and α_R accounts for all non-heating-related energy usage. Now you would like to use data from last year's logs to identify the least-squares solution for the parameters of this new model $\vec{a} = \begin{bmatrix} \alpha_T \\ \alpha_W \\ \alpha_R \end{bmatrix}$.

$\mathbf{x_T}$ (°C)	$\mathbf{x_W}$ (m/s)	E (kWh)
25	4	300
20	5	220
10	1	250
15	5	280
12	9	350

Table 2: Selected energy/weather data logs.
(x_T is the temperature, x_W is the wind speeds, and E is the energy usage.)

Provided the data in Table 2 above, explicitly set up the linear system $\mathbf{D}\vec{a} = \vec{E}$ needed to solve for the least-squares solution of the parameter vector $\vec{a}$. This means you must write out all of the matrix $\mathbf{D}$ and vector $\vec{E}$ elements for full credit.

Note: you are not asked to actually find the least-squares solutions for this part!

Solutions:

The transcription process is quite similar to the previous sub-part, but now we must be wary of the coefficients near α (which is $(x_T - 15)^2$) and γ (all constant 1).

$$\mathbf{D}\vec{a} = \vec{E} \quad \longrightarrow \quad \begin{bmatrix} 100 & 4 & 1 \\ 25 & 5 & 1 \\ 25 & 1 & 1 \\ 0 & 5 & 1 \\ 9 & 9 & 1 \end{bmatrix} \begin{bmatrix} \alpha_T \\ \alpha_W \\ \alpha_R \end{bmatrix} = \begin{bmatrix} 300 \\ 220 \\ 250 \\ 280 \\ 350 \end{bmatrix}. \quad \square$$

(c) (3 points) Now that you spent some time playing around with different prediction models, you have finally developed a surprisingly simple model (with only 3 parameters) that accurately fits last year's the energy and weather data.

Your next task is to test the model against this year's weather and energy data. Unfortunately the latest data (matrix **D** below) keeps causing your code to crash! Checking the data, you try tweaking a single entry (denoted by η) in **D** and re-running. Lo and behold... it works!

$$\mathbf{D} = \begin{bmatrix} 30 & 25 & 5 \\ 25 & 21 & 4 \\ 10 & 6 & 4 \\ 10 & \eta & 5 \\ 20 & 15 & 5 \end{bmatrix}$$

The original error message said "Math error: Solution cannot be computed." **Explain what is likely causing this error. Then use this reasoning to identify the original value of the data entry η that causes the code to crash.**

Solutions: In order to use the least squares formula, the $\mathbf{D}^T\mathbf{D}$ matrix must be invertible. This requires the **D** to have linearly independent columns. We can observe that the entries in the third column (excluding the 4th row) are equal to column 1 - column 2. So if $\eta = 5$, then the columns are linearly dependent, which would cause error. So $\eta = 5$ was our original value that caused the code to crash.

(d) (4 points) As a final test, you apply all 3 of your developed models (from question parts a, b, c) to predict daily energy usage over the last 3 days from the most recent weather data. These models generate the following daily energy usage estimates $\hat{\vec{E}}_a$, $\hat{\vec{E}}_b$, and $\hat{\vec{E}}_c$, shown below along with the actual energy values $\vec{E}$.

$$\vec{E} = \begin{bmatrix} 1 \\ 3 \\ 0.5 \end{bmatrix} \qquad \hat{\vec{E}}_a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\vec{E}}_b = \begin{bmatrix} 0 \\ 2 \\ 1.5 \end{bmatrix} \quad \hat{\vec{E}}_c = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Compute the squared error for each of the models. Based on that, which model best agrees with the actual daily usage data?

Solutions:
In order to identify the model that produced the best estimation, we must compute the squared error for each model.

$$e_a = \|\vec{E} - \hat{\vec{E}}_a\|^2 = (1-1)^2 + (3-1)^2 + (0.5-1)^2 = 0 + 4 + 0.25 = 4.25$$

$$e_b = \|\vec{E} - \hat{\vec{E}}_b\|^2 = (1-0)^2 + (3-2)^2 + (0.5-1.5)^2 = 1 + 1 + 1 = 3$$

$$e_c = \|\vec{E} - \hat{\vec{E}}_c\|^2 = (1-2)^2 + (3-2)^2 + (0.5-1)^2 = 1 + 1 + 0.25 = 2.25$$

Model c has the lowest squared error $(e_c < e_b < e_a)$, therefore our final model c most accurately predicts future energy usage based on weather patterns. □

8. (16 points) Symmetric and PSD Matrices

The eigenvectors corresponding to distinct eigenvalues of a general matrix $\mathbf{A}$ are linearly independent, and the eigenvalues of said matrix can be any real (or even complex!) numbers. In this question we consider two special classes of matrices (used ubiquitously in machine learning) and prove some essential properties about their eigenvalues/vectors.

(a) (8 points) A *symmetric* matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a square matrix such that $\mathbf{A} = \mathbf{A}^T$. Let $\vec{u}$ and $\vec{v}$ be eigenvectors of symmetric matrix $\mathbf{A}$ with distinct eigenvalues λ and μ respectively. **Show that the eigenvectors $\vec{u}$ and $\vec{v}$ are *orthogonal*.**

Hint: Consider the expression $\vec{v}^T \mathbf{A} \vec{u}$.

Solutions: By definition, $\mathbf{A}\vec{u} = \lambda \vec{u}$ and $\mathbf{A}\vec{v} = \mu \vec{v}$.
Following the hint, we see that $\vec{v}^T \mathbf{A}\vec{u} = \vec{v}^T(\lambda \vec{u}) = \lambda \vec{v}^T \vec{u}$.
Using the fact that $\mathbf{A}$ is symmetric, it also follows that $\vec{v}^T \mathbf{A}\vec{u} = \vec{v}^T \mathbf{A}^T \vec{u} = (\mathbf{A}\vec{v})^T \vec{u} = (\mu \vec{v})^T \vec{u} = \mu \vec{v}^T \vec{u}$.
Hence, $\lambda \vec{v}^T \vec{u} = \mu \vec{v}^T \vec{u}$.
Since we assume that $\lambda \neq \mu$, this implies that $\vec{v}^T \vec{u} = 0$, as desired.

The above result implies that if a symmetric matrix has all distinct eigenvalues, its eigenvectors form an orthogonal basis for $\mathbb{R}^n$. In fact, this result can still hold for the case of non-distinct eigenvalues. This is true because two equal eigenvalues (for example) correspond to a single 2D eigenspace, which you can choose to be spanned by orthogonal vectors!

(b) (8 points) A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called *positive semi-definite* if $\vec{x}^T \mathbf{A} \vec{x} \geq 0$ for any $\vec{x} \in \mathbb{R}^n$. Assume that the eigenvalues of any symmetric matrix $\mathbf{A}$ are real. **Show that a symmetric positive semi-definite matrix A has all non-negative eigenvalues (i.e, $\lambda \geq 0$).**

Hint: Apply definitions of eigenvectors, eigenvalues, and positive semi-definiteness. You will NOT need to use the fact that $\mathbf{A}$ is symmetric; this just ensures that the eigenvalues of $\mathbf{A}$ are real, which you do not need to prove.

Solutions: Let $\vec{v}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ.
Then, $\mathbf{A}\vec{v} = \lambda \vec{v}$.
Since $\mathbf{A}$ is positive semi-definite, $\vec{v}^T \mathbf{A}\vec{v} \geq 0$.
Also, $\vec{v}^T \mathbf{A}\vec{v} = \vec{v}^T(\lambda \vec{v}) = \lambda \vec{v}^T \vec{v} = \lambda ||\vec{v}||^2$.
Therefore, $\lambda ||\vec{v}||^2 \geq 0$.
Also, $||\vec{v}|| \geq 0$ since norms are non-negative.
This implies that $\lambda \geq 0$, as required.