EECS 16A Designing Information Devices and Systems I
Spring 2023 Midterm 1

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last name) (first name) (signature)

PRINT the time of your discussion section and your GSI(s) name: ________________________________

PRINT the student IDs of the person sitting on your right: ________________ and left: ________________

General Notes

- This exam has a combination of multiple choice, fill-in-the-blank, and free-response questions.
- **You must adhere to the following format to receive full credit:**
  - For fill in the blank questions, **legibly write your final answer entirely in the provided boxes**.
  - For questions with **circular bubbles**, select exactly *one* choice, by filling the bubble ●.
    ○ You must choose either this option. ○ Or this one, but not both!
  - For questions with **square boxes**, you may select *multiple* choices, by filling the squares ■.
    ☐ You could select this choice. ☐ You could select this one too!
  - For free-response questions please show your work in the provided empty boxes. Doing so enables us to reward you partial credit where applicable.

Problem Number	Problem Name	Points
2	The Warmup	14
3	Oski at the Store	14
4	Matrix Spaces	10
5	Single-pixel Camera	12
6	Shearing Sheep	14
7	Water Transitions	16
8	Building Spaces	10
9	Some Proofs	10
	Total	100

1. HONOR CODE

Please read the following statements of the honor code, and sign your name (you don't need to copy it).

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my unlimited printed resources.*
- *I did not collaborate with any other human being on this exam.*

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2. The Warmup (14 points)

(a) (6 points) In each of the following parts you are given an $m \times n$ matrix $\mathbf{A}$ and an $l \times p$ matrix $\mathbf{B}$. For each of the following subparts:

1) State the dimensions of $\mathbf{A}$ and $\mathbf{B}$. That is, provide values for m, n, l, and p

2) State whether it is possible to multiply $\mathbf{A}$ by $\mathbf{B}$ (i.e., is $\mathbf{AB}$ valid?). If so, write down the resulting product $\mathbf{AB}$.

(i).

$$\mathbf{A} = \begin{bmatrix} 0 & 4 \\ 3 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 3 \\ 2 & 3 \end{bmatrix}$$

$m =$ $\boxed{}$ $\qquad$ $l =$ $\boxed{}$

$n =$ $\boxed{}$ $\qquad$ $p =$ $\boxed{}$

Is $\mathbf{AB}$ valid? If so, compute it.

(ii).

$$\mathbf{A} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$m =$ $\boxed{}$ $\qquad$ $l =$ $\boxed{}$

$n =$ $\boxed{}$ $\qquad$ $p =$ $\boxed{}$

Is **AB** valid? If so, compute it.

(b) (2 points) For each of the following matrices, write the transpose.

(i).

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 & 6 \\ 3 & 2 & 2 & 8 \end{bmatrix}$$

(ii).

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(c) (6 points) For each of the following functions f, state whether the function is linear or not linear. If it is linear, explicitly show that it satisfies superposition and homogeneity. If it is not linear, show that it fails to satisfy either superposition or homogeneity.

(i). $f(x) = 2(x+2)$

(ii). $f(x) = 2x + x$

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3. Oski at the Store (14 points)

Oski works at the Cal Student Store and is trying to figure out how heavy various items are. Unfortunately, he cannot measure each item individually. Instead, Oski can only weigh boxes that have a mixture of different items within them. Specifically, Oski has 3 boxes:

- The first box has 1 hoodie, 1 mug, and 2 textbooks; it weighs 8 lbs.
- The second box has 2 hoodies, 3 mugs, and 4 textbooks; it weighs 17 lbs.
- The third box has 4 hoodies, 5 mugs, and 8 textbooks; it weighs 33 lbs.

Oski is interested in knowing the individual weights of hoodies (h), mugs (m), and textbooks (t).

(a) (6 points)

(i). Formulate the problem as a matrix-vector equation $\mathbf{A}\vec{x} = \vec{b}$, where each entry of $\vec{x}$ corresponds to the individual weight of an item (i.e., hoodie h, mug m, and textbook t). That is, fill out the entries for $\mathbf{A}$ and $\vec{b}$ below such that $\mathbf{A}\vec{x} = \vec{b}$.

$$\mathbf{A} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \qquad \vec{\mathbf{x}} = \begin{bmatrix} h \\ m \\ t \end{bmatrix} \qquad \vec{\mathbf{b}} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

(ii). Use Gaussian elimination to solve the system. If there are no solutions, say so. If there are infinite solutions, write the general form. Can Oski uniquely determine the individual weight of every item?

(b) (4 points) Oski's friend Nathan is using a *different* set of boxes to find the weights of hoodies, mugs and textbooks. He construct the following augmented matrix to solve for $\vec{x} = \begin{bmatrix} h \\ m \\ t \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 3 & 4 & 6 & 20 \end{array}\right]$$

Select the correct statement about Nathan's system. Justify your response in the space below using Gaussian elimination.

- ◯ The system has one unique solution.
- ◯ The system has infinite solutions.
- ◯ The system has no solutions.

(c) (4 points) Now, completely separate from the previous parts, Oski is trying to measure the weights of coats (c) and teddy bears (t) using the same strategy. This time he has two boxes:

- The first box has 1 coat and 3 teddy bears; it weighs 5 lbs.
- The second box has 5 coats and 1 teddy bear; it weighs 11 lbs.

This system is represented by the matrix-vector equation $\mathbf{A}\vec{x} = \vec{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} \qquad \vec{\mathbf{x}} = \begin{bmatrix} c \\ t \end{bmatrix} \qquad \vec{\mathbf{b}} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

Find the inverse of $\mathbf{A}$ and use it to solve for $\vec{x}$.

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4. Matrix Spaces (10 points)

Consider the following $n \times m$ matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ | & | & & | \end{bmatrix}$$

where $\vec{a}_1, \ldots, \vec{a}_m \in \mathbb{R}^n$ represent the columns of $\mathbf{A}$ and $m \geq n$.

(a) (2 points) What are the minimum and maximum possible values for the dimension of the **column space** of $\mathbf{A}$?

minimum = ☐ maximum = ☐

(b) (2 points) Select all of the *True* statements below.

☐ The dimension of the column space is **maximum** when $\vec{a}_1, \ldots, \vec{a}_m$ are linearly *dependent*.

☐ The dimension of the column space is **maximum** when $\vec{a}_1, \ldots, \vec{a}_m$ spans $\mathbb{R}^n$.

☐ The dimension of the column space is **minimum** when $\vec{a}_1 = \ldots = \vec{a}_m = \vec{0}$.

☐ The dimension of the column space is **minimum** when $\vec{a}_1, \ldots, \vec{a}_m$ are linearly *independent*.

(c) (2 points) What are the minimum and maximum possible values for the dimension of the **null space** of $\mathbf{A}$?

minimum = ☐ maximum = ☐

(d) (2 points) Select all of the *True* statements below.

☐ The dimension of the null space is **maximum** when $\mathbf{A}\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^m$.

☐ The dimension of the null space is **maximum** when $\vec{a}_1, \ldots, \vec{a}_m$ form a basis for $\mathbb{R}^n$.

☐ The dimension of the null space is **minimum** when the dimension of the column space is maximum.

☐ The dimension of the null space is **minimum** when the dimension of the column space is minimum.

(e) (2 points) Select the conditions which *must* be true if $\vec{a}_1, \ldots, \vec{a}_m$ is a basis of $\mathbb{R}^n$.

- ☐ $m = n$
- ☐ $m > n$
- ☐ $\vec{a}_1, \ldots, \vec{a}_m$ are linearly *dependent*
- ☐ $\vec{a}_1, \ldots, \vec{a}_m$ spans $\mathbb{R}^n$
- ☐ $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution for *any* $\vec{b} \in \mathbb{R}^n$.

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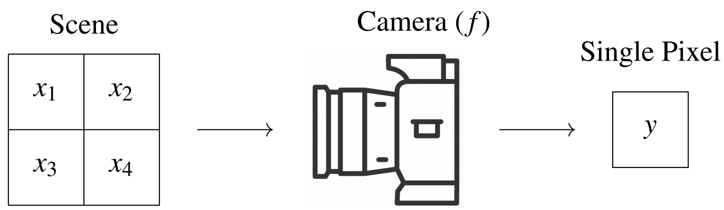
5. Single-pixel Camera (12 points)

Consider imaging a scene with a single pixel camera, as shown below.

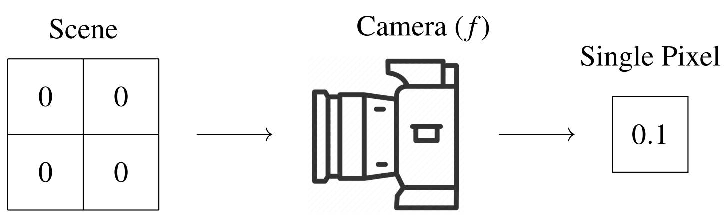


We can model the scene as a 2x2 grid of light intensity values, (x_1, x_2, x_3, x_4), and the single pixel image with y. They are related by a function f, which represents the effects of the camera, such that:

$$y = f(x_1, x_2, x_3, x_4)$$

.

(a) (3 points) Suppose you take the camera into a pitch black room (with no light), such that the scene is all 0s (i.e., $x_1 = x_2 = x_3 = x_4 = 0$). When you take an image of the scene, you a pixel value of $y = 0.1$.



(i). Could f be linear?

◯ Yes

◯ No

(ii). If yes, provide an example of a linear f such that $y = f(x_1, x_2, x_3, x_4)$. If no, provide a brief justification for why not.

(b) (3 points) Now suppose you are given a new camera and move to a new area that has light. The function representing this new camera is g, which means $y = g(x_1, x_2, x_3, x_4)$. You are told that g **is a linear function**, which means it can be written as a linear combination of its inputs (i.e., $g = ax_1 + bx_2 + cx_3 + dx_4$).

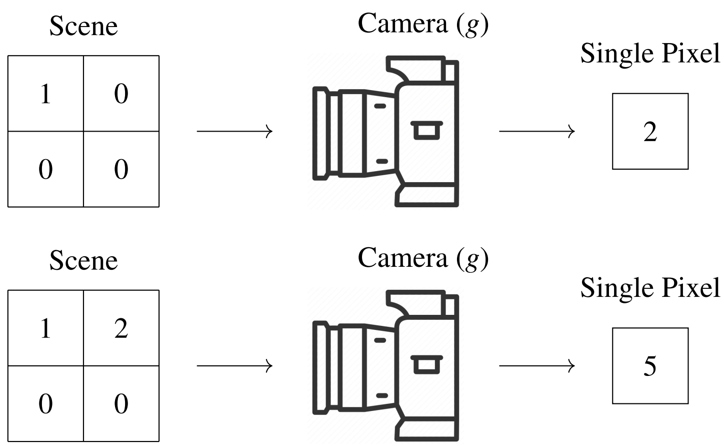
How many different scenes would you need image in order to solve for the constants a, b, c, d? Assume there is no redundancy between different scenes.

Note that you do not need to actually solve for anything, just tell us how many different scenes we would need image in order to completely specify g.

number of scenes = $\boxed{}$

(c) (4 points)

Now suppose you use this new camera, represented by the linear function g, to take two images, each of a different scene shown below. You are given the output pixel value y for each of these scenes.



For the following scenes, fill in the corresponding output value y in the box labeled Single Pixel. Show your work.

i. **Scene 1:**

Scene

2	2
0	0

$\longrightarrow$

Camera (g)

$\longrightarrow$

Single Pixel

Space for your work.

ii. **Scene 2:**

Scene

1	6
0	0

$\longrightarrow$

Camera (g)

$\longrightarrow$

Single Pixel

Space for your work.

(d) (2 points) Why were you able to solve for the missing y in part c) despite having only measured 2 scenes?

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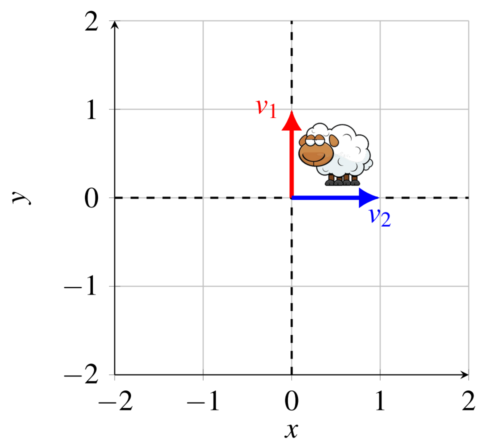
6. Shearing Sheep (14 points)

In the following questions we will consider transforming an image with a matrix. Recall that an image can be thought of as a collection of points v_i. When we transform an image with a matrix $\mathbf{A}$, we are mapping each v_i to a corresponding $u_i = \mathbf{A}v_i$. This new collection of points, u_i, comprises the transformed image. For each image below, we highlight two out of many points in its collection with the vectors labeled v_1 and v_2.

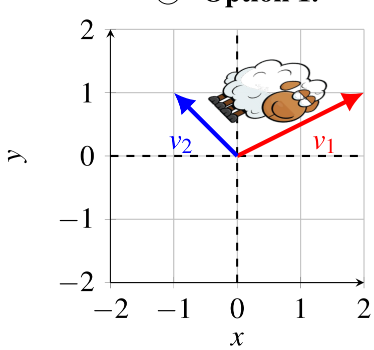
(a) (4 points) First consider the image shown below and the matrix transformation $\mathbf{A} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$



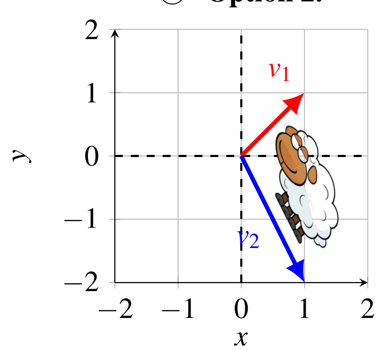
Recall that $\vec{v}_1$ and $\vec{v}_2$ lie on the image. In this case, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(i). Which of the following represents the result of transforming the given image with the matrix $\mathbf{A}$? Fill in the bubble for either Option 1, Option 2, or Option 3.

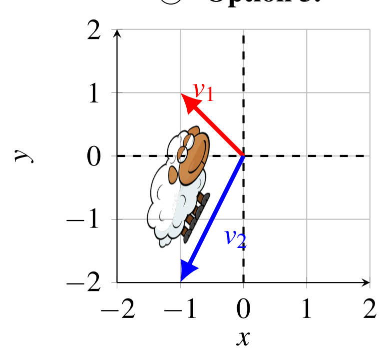
○ **Option 1.**



○ **Option 2.**



○ **Option 3.**

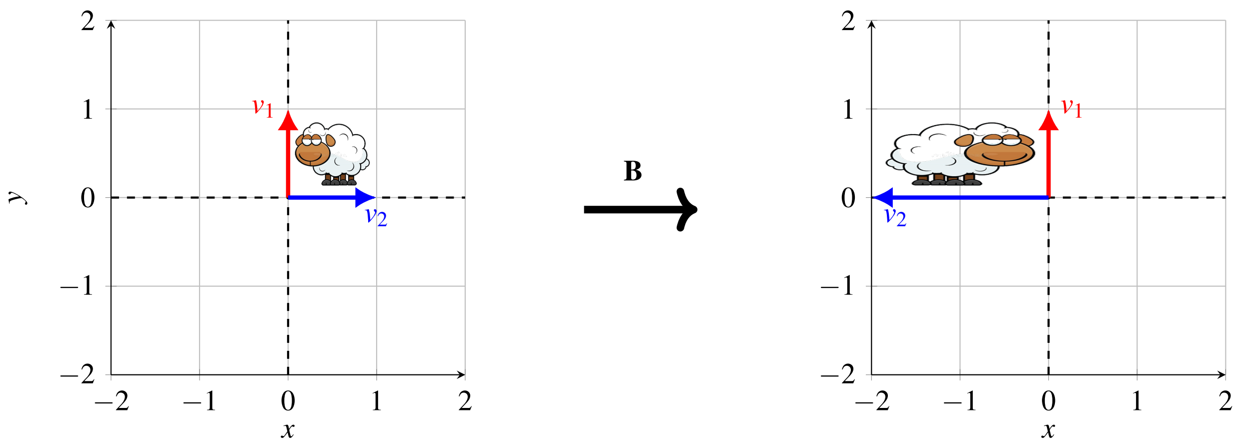


Space for your work

(ii). Is the above transformation reversible? Justify your answer mathematically.

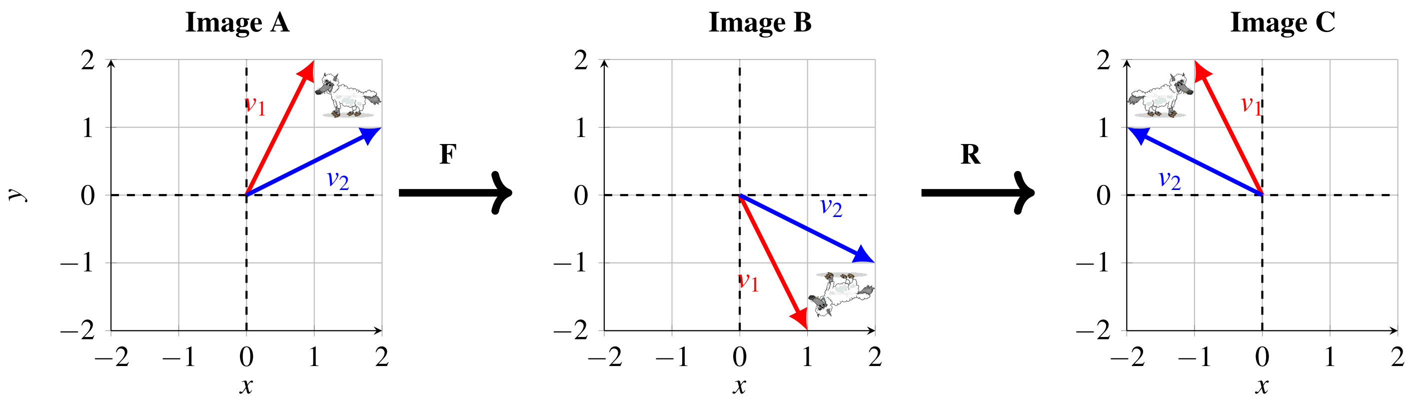
(b) (4 points)

Derive the matrix **B** which results in the below transformation.



$$\mathbf{B} = \begin{bmatrix} & \\ & \end{bmatrix}$$

(c) (6 points) Now suppose we perform a sequence of transformations on a new image (shown below as Image A). Beginning Image A, we first apply a transformation $\mathbf{F}$ to get the middle image, Image B, and then apply another transformation with matrix $\mathbf{R}$ to get the rightmost image, Image C.



Specifically, $\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects the image about the x-axis and $\mathbf{R} = \begin{bmatrix} \cos(180^\circ) & -\sin(180^\circ) \\ \sin(180^\circ) & \cos(180^\circ) \end{bmatrix}$ rotates the image counter-clockwise by 180°:

(i). Let $\vec{v}$ be a point on Image A and let $\vec{d}$ be the corresponding transformed point on Image C (after undergoing both transformations). Write d in terms of $\vec{v}$, $\mathbf{F}$, and $\mathbf{R}$.

(ii). Consider an arbitrary sequence of matrix transformations (not the F and R shown above). *In general*, will the order of transformations affect the final transformed image? That is, does it matter which matrix I apply first?

- ◯ Yes, the order of transformations will change the final image.
- ◯ No, the order of transformations will *not* change the final image.

Explain your answer using matrix properties.

(iii). Now consider the *specific* transformations F and R given above in the problem statement. Will switching the order of these transformations (i.e., applying rotation R first, then reflection F) result in the same final transformed image (Image C)?

- ◯ Yes, the order of F and R changes the final image.
- ◯ No, the order of F and R does *not* change the final image.

Justify your response mathematically.

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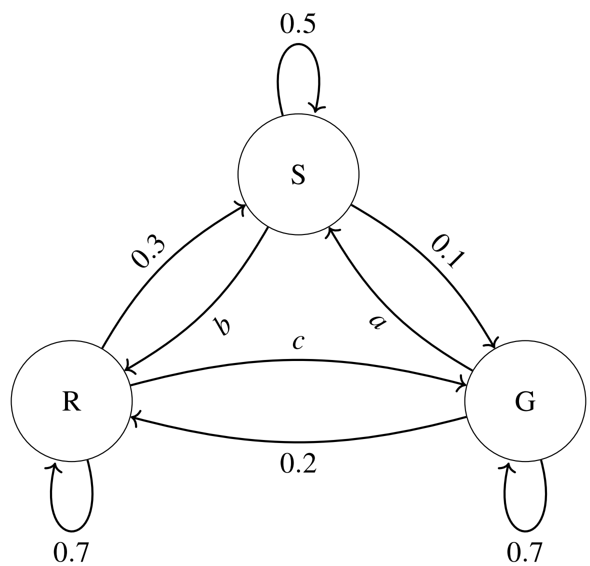
7. Water Transitions (16 points)

With all the recent (and desperately needed) rain in California, Aniruddh is interested in modeling the flow of water between different forms of water storage: snowpack (S), reservoir (R), and groundwater (G). Below is a diagram showing his state transition model for the change in water distribution.

Note that some of the transition values are symbolic (e.g., a, b, c)



(a) (4 points)

Let $\vec{m}[t] = \begin{bmatrix} m_s[t] \\ m_r[t] \\ m_g[t] \end{bmatrix}$ where $m_s[t], m_r[t], m_g[t]$ represent the volume of water in S (snowpack), R (reservoir), and G (groundwater) at time t.

(i). Fill in the below matrix $\mathbf{A}$ such that $\vec{m}[t+1] = \mathbf{A}\vec{m}[t]$. Your answer may contain a, b, and c.

$$\mathbf{A} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

(ii). What values of a, b and c would make this system conservative?

a = ☐ b = ☐ c = ☐

(b) (6 points) Anish also takes a look at the problem, and comes up with a different model and directly provides you with the following state transition matrix. $\mathbf{B} = \begin{bmatrix} 4/5 & 1/5 & 3/10 \\ 1/10 & 7/10 & 1/10 \\ 1/10 & 1/10 & 3/5 \end{bmatrix}$

(i). Does a steady-state exist for this model?

◯ Yes

◯ No

(ii). If so, define the set of steady-state vectors. If not, why not?

(c) (6 points) Finally, Vivian also comes up with a model of her own. However, she does not directly give you a state transition matrix. Instead, she tells you the following information:

- The eigenvalues of her transition matrix, $\mathbf{C}$ are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = \frac{1}{2}$.
- The eigenvectors corresponding to these eigenvalues are:
  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix},$

Suppose you know the current state of the system (at time t) is $\vec{m}[t] = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}$. Compute the previous state, $\vec{m}[t-1]$.

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8. Building Spaces (10 Points)

(a) (4 points) What is the nullspace of the following matrix?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

(b) (6 points) Assume for a system of equations $A\vec{x} = b$, we find that the set of solutions for $\vec{x}$ can be written as

$$X = \left\{ \vec{x} \,\middle|\, \vec{x} = \vec{c} + \alpha \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

You want to pick a value of $\vec{c}$ that will ensure that the set of solutions X forms a vector subspace of $\mathbb{R}^3$. For each of the following possible values for $\vec{c}$, select whether it would make X a subspace of $\mathbb{R}^3$. Justify your answer by showing whether or not it satisfies the 3 properties of a subspace.

i. $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

◯ X is a subspace

◯ X is *not* a subspace

ii. $\vec{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

◯ X is a subspace

◯ X is *not* a subspace

iii. $\vec{c} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$

◯ X is a subspace

◯ X is *not* a subspace

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9. Some Proofs (10 points)

(a) (5 points) Prove that if there are two unique solutions $\vec{x}_1$ and $\vec{x}_2$ to the system $A\vec{x} = \vec{b}$, then there are infinitely many solutions $\vec{x}$ to this system.

(b) (5 points) A matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ has eigenvalues λ_1, λ_2, ... λ_n and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Let $\lambda_1 = 0$, $\lambda_2 = 0$ and λ_3, λ_4, ... $\lambda_n \neq 0$. Prove that $\vec{x} \in Null(A)$ for any $\vec{x} \in span\{\vec{v}_1, \vec{v}_2\}$.