EECS 16A Designing Information Devices and Systems I
Spring 2015 Homework 2

This homework is due February 12, 2015 at 5PM.

1. **Transformation Matrices**

   Download the file `Homework2.ipynb` and follow the instructions to define rotation, scaling and translation matrices.

2. **Roots of unity**

   As we will see later on in this class, complex numbers are often used to represent signals we wish to communicate wirelessly. One of the reasons is that complex numbers can be easily related to sine and cosine waves using Euler's formula. In this problem, we will therefore explore some properties of vectors with complex numbers. In particular, these vectors play an important role in something called a "Fourier transform" that we will learn about in EE16B.

   (a) Recall Euler's formula
   $$e^{i\theta} = \cos\theta + i\sin\theta$$
   Recall, $i = \sqrt{-1}$. Derive Euler's formula using calculus or Taylor series.

   (b) Using Euler's formula, show that $e^{2\pi i \frac{k}{N}}$, $0 \le k \le N-1$ is a N^{th} root of unity (a number z such that $z^N - 1 = 0$). For $N = 5$ draw these (by hand is fine) on the complex plane.

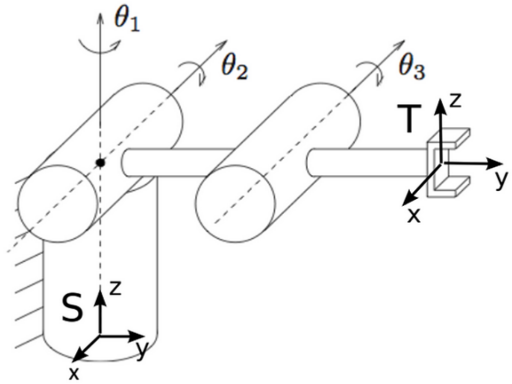
   (c) Calculate $\sum_{k=0}^{N-1} e^{2\pi i \frac{k}{N}}$.

   (d) Consider the set of vectors $\{\mathbf{v}_k\}$, with $0 \le k \le N-1$, where $\mathbf{v}_k = \begin{bmatrix} e^{2\pi i \frac{k}{N}\times 0} & e^{2\pi i \frac{k}{N}\times 1} & e^{2\pi i \frac{k}{N}\times 2} \dots e^{2\pi i \frac{k}{N}\times (N-1)} \end{bmatrix}^T$.
   Thus, each element of the vector is a term of the form $e^{2\pi i \frac{k}{N}\times m}$ where $0 \le m \le N-1$. Calculate the inner product $(\mathbf{v}_l^T)^*\mathbf{v}_j$ where $\mathbf{v}_l$ is the vector $\mathbf{v}_k$ when $k = l$ (l and j are variables).

   (e) Consider a vector in $\mathbb{R}^3$, $X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. Find the set $\{\alpha_0, \alpha_1, \alpha_2\}$ such that $\sum_i \alpha_i \mathbf{v}_i = X$. Thus, show that $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ form a basis for $\mathbb{R}^3$.
   Hint: the coefficients α_i, if stacked together as a vector α, are the solutions to the matrix-vector equation $\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \alpha = X$, where the first matrix is composed of column vectors $\mathbf{v}_k$ evaluated at $k = 0, 1, 2$.

3. **Vector spaces**

   Vector spaces are often used to represent the set of possible actions in a space. For instance, a robotic arm might have only directions of motion that are restricted in a particular way because of the mechanics of the design. Take a robotic arm that can only extend its length and turn around the z-axis (θ_1 in the diagram below). The arm of the robot is limited to the xy-plane, which is defined by $\left\{ \begin{bmatrix} x & y & z \end{bmatrix} : x, y, z \in \mathbb{R}, z = 0 \right\}$. If we add another degree of freedom by adding a hinge that lets the arm rotate about θ_2, we are adding a dimension to the range of motion of the arm.



(a) Suppose we design a robot arm, iArmEE 16.0, that has an extendable arm that is physically built such that its arm can only rotate around an axis defined by the vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ instead of rotating around the z-axis. Come up with a mathematical model (plane equation) that defines the positions iArmEE 16.0 can reach.

(b) True or False: The mathematical model you defined above forms a vector space. If it forms a vector space, can you find a basis?

(c) Now suppose a new model, iArmEE 16.1, has a fixed stand. From the base of the robot (point of origin), there is a fixed-length stand of length 1 unit in the z-axis direction (the vertical cylinder in the diagram above). The original arm is then mounted on this stand, which can be modelled by the vector $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Thus, the range of motion of the arm from part (a) is translated by this vector. Formulate a new mathematical model (plane equation) that defines the positions iArmEE 16.1 can reach.

(d) True or False: The mathematical model you defined above forms a vector space. If it forms a vector space, can you find a basis?

Subspaces: Given a vector space $(\mathbb{V}, \mathbb{F})$, a non-empty subset $\mathbb{W}$ of $\mathbb{V}$ that is closed under addition and multiplication, such that $\mathbb{W}, \mathbb{F})$ is a vector space itself, is called a subspace of $(\mathbb{V}, \mathbb{F})$.

Consider now a different robot that can move along straight line tracks in a plane. We can add new tracks and this allows the robot to explore new places. Each new track is a line. Let us say that the entire "possible" set of motions of the robot are captured by a plane, say $\mathbb{R}^2$. We know that $(\mathbb{R}^2, \mathbb{R})$ forms a vector space. Each of the tracks is a subspace of the plane.

(e) The effective subspace of motion from all the tracks together would be captured by the union of the tracks. However, adding the motions along two tracks (lines) does not lead to a vector space, and in fact, it is intuitively clear if you only could move along two particular lines in the plane, you cannot reach all points in the plane. (Think about trying to get to every address in Berkeley but only driving on University and Shattuck.) This principle can be generalized.

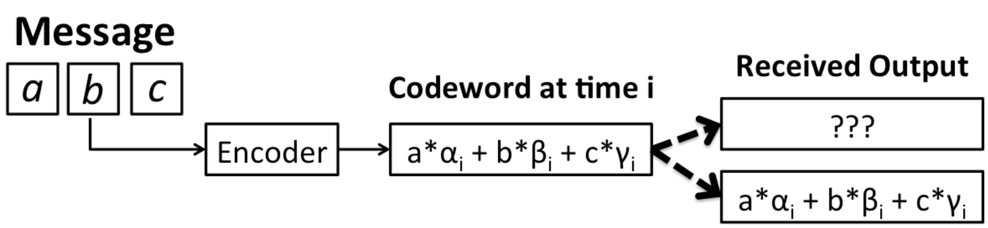
Prove that the union of two subspaces, $(\mathbb{A}, \mathbb{F})$ and $(\mathbb{B}, \mathbb{F})$ of the vector space of $(\mathbb{V}, \mathbb{F})$ is a subspace of $(\mathbb{V}, \mathbb{F})$ if and only if one of the subspaces is contained in the other. (So in the 2-D example, the union of two tracks would make a subspace if and only if the two tracks are exactly on top of each other.)

4. Fountain codes

Consider a sender, Alice, and a receiver, Bob. Alice wants to send a message to Bob, but she can't physically send the message all at once. Instead, she breaks her message up into little packets (say, 8 bits long) which she sends across a wireless channel (think radio transmitter to antenna.) She knows some of the packets will be dropped or erased along the way (birds sitting on the telephone wires again...), so she needs a way to encode her message. This way, even if Bob gets a message missing parts of words he can still figure out what Alice is trying to say! One coding strategy is to use Fountain Codes. Fountain codes are a type of error-correcting codes based on principles of Linear Algebra. They were actually developed right here at Berkeley! The company that commercialized them, Digital Fountain, (started by a Berkeley grad, Mike Luby), was later acquired by Qualcomm. In this problem, we will explore some of the underlying principles that make Fountain codes work in a very simplified setting.

In this problem, we concentrate on the case with transmission erasures, where the bad transmission causes some of the message to be corrupted. Let us say Alice wants to convey the set of her three favorite ideas covered in 16A lecture each day to Bob. For this, she maps each idea to a real number and wants to convey the 3-tuple $\begin{bmatrix} a & b & c \end{bmatrix}$ (Let us say there are an infinite number of ideas covered in 16A). At each time step, she can send one number, which we will call a "symbol" across. So one possible way for her to send the message is to first send a, then send b, and then send c. However, this method is particularly susceptible to losses. For instance, if the first symbol is lost, then Bob will receive $\begin{bmatrix} ? & b & c \end{bmatrix}$, and he will have no way of knowing what Alice's favorite idea is.

(a) The main idea in coding for erasures is to send redundant information so that we can recover from losses. So if we have three symbols of information, we might transmit six symbols for redundancy. One of the most naive codes is a called the repetition code. Here Alice would transmit $\begin{bmatrix} a & b & c & a & b & c \end{bmatrix}$. How much erasure can this transmission recover from? Are there specific patterns it cannot handle?



(b) A better strategy for transmission, is actually to send linear combinations of symbols. Alice and Bob decide in advance on a "codebook", i.e. a set of vectors $\mathbf{v}_i = \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \end{bmatrix}^T$, $1 \le i \le 6$. At time i, Alice transmits $k = \begin{bmatrix} a & b & c \end{bmatrix} \mathbf{v}_i = a\alpha_i + b\beta_i + c\gamma_i$. How can you write the transmitted symbols as a matrix multiplication in this case?

(c) What are the vectors $\begin{bmatrix} \alpha_i & \beta_i & \gamma_i \end{bmatrix}^T$, $1 \le i \le 6$ that generate the repetition code strategy in part (a)?

(d) Suppose now we choose a codebook with seven vectors $\mathbf{v_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{v_2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $\mathbf{v_3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $\mathbf{v_4} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\mathbf{v_5} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$, $\mathbf{v_6} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, $\mathbf{v_7} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. Again, at time i, Alice transmits $k = \begin{bmatrix} a & b & c \end{bmatrix} \mathbf{v}_i$. In this case, what pattern of losses let Bob recover the message?

(e) Suppose, using the codebook in part (d), Bob receives $\begin{bmatrix} 6 & ? & ? & 2 & 3 & ? & ? \end{bmatrix}$. What was the transmitted message? Express the problem as a system of linear equations using matrix multiplication.

(f) Fountain codes build on these principles. The basic idea used by these codes is that Alice keeps sending linear combinations of symbols until Bob has received enough to decode. So at time 1, Alice sends the linear combination using $\mathbf{v_1}$, at time 2 she sends the linear combination using $\mathbf{v_2}$ and so on. After each new linear combination is sent, Bob sends back an acknowledgement to tell Alice whether or not he can decode her message. So clearly, the minimum number of transmissions for Alice is 3. If Bob receives the first three linear combinations that are sent, Alice is done in three steps! But because of erasures, he might not. Suppose Alice used $\mathbf{v_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{v_2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $\mathbf{v_3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ as her first three vectors, but she has still not received an acknowledgement from Bob. Should she choose new vectors according to the strategy in part (a) or the strategy in part (d)? Why?