EECS 16A Designing Information Devices and Systems I
Spring 2015

Homework 8

This homework is due April 16, 2015 at 5PM.

1. **Tomography**

   Underdetermined systems of equations can be very useful in medical imaging contexts. Often parts of the body are imaged to find abnormalities in the tissue. The technique to do this is called tomography (we briefly talked about this in lecture). Tomography uses the fact that different materials in the body absorb different amounts of light. Readings of the absorption of beams of light from various directions onto the tissue are used to reconstruct the image of the tissue.

   We won't be developing an exact parallel to tomography here, but we'll consider a caricature model. Consider an unknown vector $\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, that represents a tissue image with $n = 10^8$ pixels. Let us assume that if the tissue is normal and healthy — which should be the case for the large majority of the pixels in the image — the absorption of light is essentially equal to 0. Let us further assume that if there is e.g. a blood clot (or some other abnormality) in the tissue, the absorption value is very high.

   Because each absorption measurement may take anywhere from several ms to even several seconds in order to achieve reasonable accuracy, it is typically infeasible to get 10^8 or more measurements in order to estimate the value of every pixel in the image using a technique like least squares. Because of this, we would like to use the fact that number of blood clots if very low, and thus the number of non-zero entries within the vector x should also be very low to then approximate the image using fewer measurements.

   Although we can't measure the entire tissue image directly, let us say that we are able to get 10^3 measurements of the total absorption for a volume of tissue that spans multiple pixels in the original image. In other words, the result of each absorption measurement b_i is equal to the dot product of a "sampling vector" a_i (note that a_i is a column vector) and the image vector x. For each measurement that is taken, each pixel value is either recorded or not. Therefore, the row vectors $\vec{a}_i$ are binary.

   To understand what is going on, let us consider a small example below. Here we take three measurements for a 2×2 image, i.e. $n = 4$ and $m = 3$. Therefore, $\vec{x}$ is 4×1, and $i = 3$.

a	c
b	d

   Figure 1: Image

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix}^T = a + b \tag{1}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix}^T = b + c + d \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix}^T = a + d \tag{3}$$

(a) Set up an underdetermined system of equations for the case of 10^3 measurements for 10^8 pixels. Give the form of a minimum-norm solution.

(b) Download `prob8.ipynb`. Find the minimum norm estimate for the images using the data in iPython files `phantom1.mat` and `phantom2.mat`.

2. Military and civilian GPS

The Global Positioning System (GPS) is a space-based satellite navigation system that provides location and time information in all weather conditions, anywhere on or near the Earth where there is an unobstructed line of sight to four or more GPS satellites. The last homework explored a set of questions relating identifying which GPS satellite was transmitting a particular GPS signal. Using Gold codes and autocorrelation, we were able to identify which satellite was transmitting even though all the satellites were transmitting at the same frequency using a spread-spectrum technology.

This problem will explore a different way of separating signals that are simultaneously transmitted at the same frequency. This strategy is actually also used in the GPS system for separating between the civilian and military GPS systems. The last homework considered a sampled version of the signal transmitted by the GPS satellite, as in, we assumed that the continuous time signal $x(t)$ transmitted by the satellite was converted to a discrete time sequence $x[n]$ using sampling (more about this in the coming lectures!) Here we will explore the signal in the raw continuous time format.

The GPS signal that you use on your phone is part of the civilian GPS system.

Some of the GPS satellites are used for transmitting both the civilian and military GPS signals, and they are transmitted *at the same frequency*! Despite this, your phone is able to remove the interference from the military grade signal (since the military signal is encrypted, a civilian device would not be able to decode the information in the signal and it only serves as noise/harmful interference), and extract the relevant civilian signal. How can it do this? We explore this in the problem below.

The scheme uses an important property of waves: a particular frequency has two components which can carry information, the amplitude and the phase. Consider a signal $x(t) = A\cos(\omega t + \phi)$. The frequency of this signal is ω. The amplitude of the signal is A, and is the first part of the signal that can carry information. The phase of the signal is ϕ. This is the second part of the signal that can carry information.

(a) Show that $A\cos(\omega t - \frac{\pi}{2}) = A\sin(\omega t)$. What does this mean about the phase difference between a sine wave and a cosine wave?

(b) Draw out the functions $x(t) = A_1\cos(\omega t)$, $y(t) = A_2\sin(\omega t)$ and $z(t) = x(t) + y(t)$ against time by hand for some non-zero A_1 and A_2 of your choice (e.g. $A_1 = A_2 = 1, \omega = \frac{2\pi}{3}$). (Feel free to use iPython to help you draw this, but note that a program can only do discrete time plotting/calculations.)

(c) Calculate the continuous time cross-correlation between $x(t) = A_1\cos(\omega t)$ and $y(t) = A_2\sin(\omega t)$.

(d) The GPS system uses phase information to differentiate the two signals. Let $z(t) = A_1\cos(\omega t) + A_2\sin(\omega t)$ be the signal transmitted by the satellite. The civilian signal is carried on the cosine wave (at frequency $\omega = \frac{1}{2\pi}66 \times 10.23 radians/second$ but you don't need to use this.), and so the sine component of the signal is noise to you. Develop a strategy to extract the civilian signal from the sum of the two signals.

(e) What would happen if instead of $\sin(\omega t)$ the military grade signal was transmitted at $\cos(\omega t + \phi_1)$, where $\phi_1 \neq \frac{\pi}{2}$? Will your strategy still work?

3. Phasors

Provide a well-labeled hand sketch for each of the following signals. If a signal is complex-valued, treat it as the position of a particle on the complex plane and plot its trajectory, indicating by arrows the direction of motion and labeling its position at $t = 0$ and at the other salient points in time. If the signal is periodic, determine its fundamental period.

(a) $x(t) = cos(\pi t)e^{i\pi t/2}$

(b) $x(t) = e^{i(1-i\pi/2)t}u(t)$ where $u(t)$ is the continuous-time unit-step function defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(c) $x(t) = Re\{e^{(-1+i\pi/2)t}u(t)\}$

(d) $x(t) = sin[\theta(t)]u(t)$ where $\theta(t) = \pi\alpha t^2 + 2\pi f_0 t + \phi$, and $\alpha = 1, f_0 = 1, \phi = \pi/4$
Determine a reasonably simple expression for the instantaneous frequency $f(t) = \dot{\theta}(t)$ in terms of α, f_0 and ϕ. How is the $f(t)$ frequency changing with time?

4. Beat effect

Watch the video at:
http://video.mit.edu/watch/tuning-forks-resonance-a-beat-frequency-11447/.
This video demonstrates the generation of a beat frequency, which is what we will explore in this problem.

Most of us are unable to tell the difference between frequencies that are very close to each other, e.g. 552 and 546 Hz, if we were to listen to tones generated at these frequencies one after the other. However, if the sounds reach our ears simultaneously, what we hear is a sound at a frequency which is the average of the two frequencies. Furthermore, as the video demonstrates, you hear a variation in the intensity of the sound, and this variation corresponds to the difference between the two frequencies.

(a) Consider two signals $x(t) = \cos(2\pi 101t)$ and $y(t) = \cos(2\pi 99t)$ that are simultaneously received at a receiver. Calculate the frequency of the beats that you will hear.

(b) It turns out the beat effect is what police radar guns use to calculate the speed of a car. Consider a radar gun that sends out a pulse $x(t) = \cos(2\pi\omega t)$. This is incident on a car that is moving towards the radar gun (the detector) with velocity v, and reflects back at the receiver. Recall that because of the Doppler effect the reflected wave will have a shift in frequency. If the original frequency of the wave was f, the the reflected wave will have frequency $f' \approx f\frac{c}{c-v}$, where c is the speed of sound. What is the "beat frequency" that the radar gun will detect? How can you use this to detect the speed of the car?