EECS 16A Designing Information Devices and Systems I
Spring 2015 Homework 9

This homework is due April 23, 2015 at 5PM.

1. **AM in the morning**

   It's early in the morning and far too late in the semester for you to feel like getting out of bed. You roll over and try to check Facebook on your phone but for some reason you can't connect. You try email — and realize that the internet connection is down! Oh no. And you've lost all bars — no texts either! Panic sets in. What's been happening on the CS Facebook group? How will you survive if you can't get the latest Meerkat stream? Yikyak? Snapchat? It is a tragedy.

   Fortunately, at that moment your friend who is also taking 16A knocks on your door. Her phone and computer have also lost all connectivity. She has a great idea — instead of being reliant on commercial networks — why not use the techniques from class to set up an AM radio station ? That way everyone can tune into the right frequency and make sure they get the latest info on what is happening in the exciting world of Soda and Cory.

   To do this you start with getting a wireless transmitter. The transmitter uses a carrier frequency of $\cos(\omega t)$ (and you know ω). So to send your desired signal $A(t)$, you modulate it by $\cos(\omega t)$ and transmit $A(t)\cos(\omega t)$. To be able to demodulate a signal transmitted using this carrier frequency, you know you need to be able to multiply the signal by another $\cos(\omega t)$. For this, each person in the class would need another oscillator that oscillates at exactly the same frequency $\cos(\omega t)$.

   However, in the real world oscillators often end up with a phase offset that is unknown. So if all 60 students had 60 different receivers, we would end up with 60 different offsets that are unknown! This question explores how you might be able to recover the transmitted signal $A(t)$ even when you cannot demodulate using exactly $\cos(\omega t)$ but are limited to using $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$.

   (a) Phase offsets often arise as a result of delays/synchronization errors in a system. Consider the case where the clocks on the transmitter and the receiver are off by Δt seconds. So when the transmitter reads t seconds, the receiver reads $t + \Delta t$ seconds. So when the transmitter generates $\cos(\omega t)$, the receiver generates $\cos(\omega(t + \Delta t))$. If $\omega = 2\pi 100 MHz$, for what value of Δt will the transmitter and receiver be completely out of phase (i.e. have phase offset $\frac{\pi}{2}$?)

   (b) Now it is very difficult to know the exact phase offset or synchronization error between the transmitter and receiver. Assume that the receivers oscillator can only generate $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$. How can you extract the original signal $A(t)$? Assume that $A(t)$ is varying very slowly in time compared to the carrier frequency ω.

   (c) Would your strategy still work if the frequency of $A(t)$ was close to ω, i.e. if $A(t)$ changed over the timescale of a period of the signal?

2. **On-off keying**

   As we discussed in lecture, one of the simplest forms of digital (as opposed to analog) communication is "On-Off Keying", which is often abbreviated as OOK. In OOK, binary signals are sent by multiplying a signal at a carrier frequency by either 1 or 0. In this problem we'll use the frequency domain representation of the signals sent by an OOK transmitter to highlight how bandwidth and data-rate are related to each other.

To describe this more mathematically and focusing on transmitting a single bit of information, the signal transmitted in OOK is

$$v_{TX}(t) = A(t)\sin(2\pi f_c t)$$

If the data bit D is a 1, then $A(t) = 1$ for $0 \le t \le T_{\text{bit}}$ and 0 elsewhere; if D is a 0, then $A(t) = 0 \forall t$.

In the above equations, T_{bit} is the duration of time over which we send each data bit. In an actual data stream with multiple data bits $D[1 \ldots n]$, we would pack a series of bits together by spacing them apart in time by one T_{bit}. Thus, $D[k]$, the k^{th} bit in the data stream, has an amplitude of $A(t - k * T_{\text{bit}})$.

(a) Assume we will send a single data bit $D[0] = 1$. The carrier frequency is $f_c = 1000\,\text{MHz}$ and $T_{bit} = 100\,\text{ns}$. Sketch the $A(t)$ that corresponds to this transmission. Also sketch $v_{TX}(t)$ for $-1\,\mu\text{s} < t < 1\,\mu\text{s}$.

(b) As we learned in class, if we know the spectrum of $A(t)$, the spectrum of $v_{TX}(t)$ will simply be a frequency-shifted version of the original spectrum of $A(t)$. Using iPython and the code provided in `prob9.ipynb`, plot the spectrum of $A(t)$ for $T_{\text{bit}} = 200\,\text{ns}, 100\,\text{ns}, 50\,\text{ns}$ and $25\,\text{ns}$. $A(t)$ is the same as in part (a).

(c) The "three-decibel bandwidth" of a signal $A(t)$ is defined as the frequency $f = f^*$ such that the magnitude of the coefficient of f^* is $\frac{1}{\sqrt{2}}$ times the magnitude of the coefficient at $f = 0$. Using iPython and the code in `prob9.ipynb`, find the three-decibel bandwidth of $A(t)$ for each of the values of T_{bit} in part b). and plot the relationship between the two values.

(d) *Bit rate* is defined as the number of bits we can send per second. Given the results of part c), how are bit rate and the required bandwidth related?

3. Compression of Music and Images

Frequency domain representations of signals are frequently used for compression. You have seen previously that the salient aspects of a vector/image can be obtained by looking only at a few components of the vector. For example, in the last homework you used the least-norm setup to estimate a vector in an underdetermined setup. Back in the eigenvalues and eigenvectors homework, you were able to reconstruct an image using only a few of it's eigenvectors. If you recall, the more eigenvectors used in the reconstruction, the better it looked.

Representing a signal in frequency domain is actually equivalent to just representing the signal in a different basis. We can build on the ideas we developed in the eigenvalues and eigenvectors section here. It turns out that we can throw out some components of the frequency representation of a signal but still keep the essential character of the signal. Depending on the characteristics of the signal different frequency components might be more important than others, and there are various ways of determining which components are more important than the others. Here, we will explore one compression mechanism.

Let us first consider audio signals. Any audio signal can be represented as a linear combination of it's constituent frequencies as you know. The audible frequencies for the human ear are in the range 20-20,000Hz. One of the methods of compressing an audio signal is to look at the frequency domain representation of the signal and drop the components that correspond to the high frequencies. This is actually what is done (to some extent) in compression formats like mp3.

(a) In the file `prob9.ipynb`, download a few seconds of the song Happy. Calculate the frequency domain representation of the song. Set a certain fraction of the highest frequencies to 0, remap to the time domain, and listen to your compressed reconstruction.

(b) Set 50, 80, 90, and 95 percent of the high frequencies to 0. What do you notice as you set more frequencies to 0? Experiment to see what fraction of high frequencies you can throw away before hearing a difference in the music.

(c) Now, we consider this in the context of images. We can treat each column of the image as a one-dimensional array (just like audio) and repeat the same exercise as above. That is, convert each column to the frequency domain, set high frequencies to 0, and reconstruct. The file `prob9.ipynb` has code to iterate through each column and apply this method. Set 50, 80, 90, and 95 percent of the high frequencies to 0. What do you notice about the final image once you throw most of the high frequencies away?

(d) Repeat the same exercise, except now treating each *row* of the image as a one-dimensional array. Set 50, 80, 90, and 95 percent of the high frequencies to 0. What are the differences in the reconstructions when the image is decomposed by rows vs. columns? Why do you think this happens?

4. Bandwidth Overlap and Interference

As discussed in lecture, one of the key characteristics of wireless communication systems is that we will have many different users or even independent systems (e.g., WiFi and Bluetooth) operating at the same time, and we would like these systems to interfere with each other's operation as little as possible.

For two purely sinusoidal signals at different frequencies, we could extract the constant amplitude of each signal by multiplying with a sinusoid at each frequency. In this problem we'll explore how to avoid interference when we use a time-varying envelope to modulate the sinusoidal carrier.

(a) Assume we have two wireless communication systems, 1 and 2. System 1 has a carrier frequency of 310 MHz, and System 2 has a carrier frequency of 311 MHz. The receiver will pick up both signals at the same time. If System 1 is transmitting a constant amplitude a_{310} and System 2 is transmitting a constant amplitude a_{311}, the signal at the receiver's antenna, $v_r(t)$, will be

$$v_r(t) \;=\; a_{310}\cos(2\pi(310\times 10^6)t) + a_{311}\cos(2\pi(311\times 10^6)t)$$

Sketch the spectrum of the received signal $V_r(f)$.

(b) Now let's consider what happens when we want to send modulated signals on each of the carrier frequencies. Assume that the amplitudes remain the same, but rather than sending a constant signal, both systems want to send a sinusoid of frequency f_{mod}. Then $v_r(t)$ is given by

$$v_r(t) \;=\; a_1(t)\cos(2\pi(310\times 10^6)t) + a_2(t)\cos(2\pi(311\times 10^6)t)$$

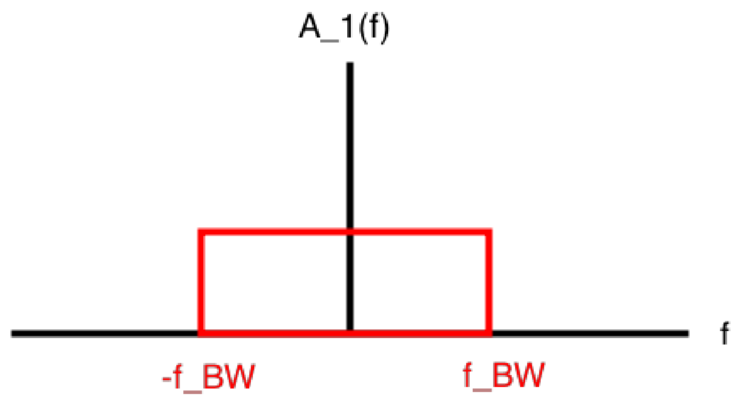
where $a_1(t)$ and $a_2(t)$ are given by

$$a_1(t) \;=\; a_{310}\cos(2\pi f_{\text{mod}} t)$$
$$a_2(t) \;=\; a_{311}\cos(2\pi f_{\text{mod}} t)$$

Sketch the spectrum of the received signal $V_r(f)$ for $f_{\text{mod}} = 100$ kHz and $f_{\text{mod}} = 200$ kHz.

(c) Assume that the receiver always extracts $A_1(t)$ by computing $v_r(t)\cos(2\pi(310 \times 10^6)t)$ and averaging this signal over an appropriately chosen time window. As f_{mod} is increased above 200 kHz, at what value of f_{mod} would we first fail to recreate $a_1(t)$ without any interference from $a_2(t)$? Redraw the spectrum $V_r(f)$ at this value of f_{mod}, and predict the output of your receiver using complex exponentials or trig identities.

(d) In reality, most of the signals we want to transmit won't look like pure sinusoids. They will instead have energy distributed over some range of frequencies (this is the so-called "bandwidth"). Suppose we want to receive $a_1(t)$. In the simplest case, we can imagine that that spectrum $A_1(f)$ looks like the picture below:



Within the bandwidth, all frequencies have the same magnitude, and outside the bandwidth, the magnitude is 0. Assume that both System 1 and System 2 would like to communicate signals that have the same bandwidth f_{BW}. Given your answer to part c), what is the maximum value of f_{BW} you can use while avoiding interference between the two systems? Sketch the spectrum $V_r(f)$ of the received signal $v_r(t)$ for a value of f_{BW} slightly larger than the maximum, and show where in the spectrum interference occurs.

5. Orthogonal Decomposition of Signals

Let x be a T-periodic, complex-valued, continuous-time signal x described by the decomposition

$$\forall t \in \mathbb{R}, \quad x(t) = X_0\varphi_0(t) + X_1\varphi_1(t) + \cdots + X_{N-1}\varphi_{N-1}(t). \tag{1}$$

The coefficients $\{X_k\}_0^{N-1}$ are, in general, complex scalars, and the functions $\{\varphi_k\}_0^{N-1}$ are T-periodic and mutually orthogonal according to the inner product

$$\langle \varphi_k, \varphi_\ell \rangle \triangleq \int_{\langle T \rangle} \varphi_k(t)\varphi_\ell^*(t)dt.$$

Specifically, $\langle \varphi_k, \varphi_\ell \rangle = 0$ for $k \neq \ell$.

(a) By projecting x onto φ_k, we can determine the coefficient X_k independently of all other coefficients. In particular, look at

$$\langle x, \varphi_k \rangle = \langle X_0\varphi_0 + X_1\varphi_1 + \cdots + X_{N-1}\varphi_{N-1}, \varphi_k \rangle,$$

and use the elementary properties of an inner product to simplify the right-hand side and obtain a reasonably simple expression for X_k.

(b) The *energy* E_x of the signal x is defined as

$$E_x \triangleq ||x||^2 \triangleq \langle x, x \rangle = \int_{\langle T \rangle} x(t)x^*(t)dt = \int_{\langle T \rangle} |x(t)|^2 dt.$$

Determine E_x in terms of the coefficients $\{X_k\}_0^{N-1}$ and the energies $\{E_{\varphi_k}\}_0^{N-1}$ of the mutually orthogonal functions $\{\varphi_k\}_0^{N-1}$.

(c) Suppose you're only allowed to estimate x by keeping M out of N terms (where $M < N$) in the orthogonal expansion of Equation (1). Let $\widetilde{x}$ denote the estimate of x, so $e = x - \widetilde{x}$ denotes the estimation error. Which M out of N terms would you keep in the orthogonal expansion of Equation (1), so that the energy E_e of the error signal is minimized?

(d) Let $\{\varphi_k = e^{i2\pi k f_0 t}\}_0^{N-1}$, where f_0 is a fundamental frequency and $T_0 = 1/f_0$ is the fundamental period of $\varphi_1 = e^{i2\pi f_0 t}$.

i. Show that $\varphi_k \perp \varphi_\ell$ for $k \neq \ell$.

ii. Determine the energy E_{φ_k} of the complex exponential signal φ_k.

iii. Let $x = \sum_{k=0}^{N-1} X_k \varphi_k(t)$ denote the frequency decomposition of the signal x. In particular,

$$x(t) = \sum_{k=0}^{N-1} X_k e^{i2\pi k f_0 t}.$$

Determine a reasonably simple expression for the coefficients X_k, where $k = 0, \ldots, N-1$.