EECS 16A　　Designing Information Devices and Systems I
Spring 2015

Note 3

Lecture notes by Christine Wang (01/27/2015)

Introduction to Vectors

Remark. Often, vectors are represented as letters in boldface(**x**), or with a small arrow on top ($\vec{x}$). In these lecture notes, vector notation is implied either by context (for example, during vector addition) or by describing the space of the vector (for example, $x \in \mathbb{R}^3$).

Suppose we are given a collection of n real numbers, $x_1, x_2, \cdots, x_n$. For example, for $n = 3$, we could have $-1, 3.5, 0$. (This vector would be in 3D space.)

We can represent this collection as a single point in an n-dimensional space, denoted:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Definition 3.1 (vector): $x \in \mathbb{R}^n$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, is called a **vector**. In this case, we say it is a vector in $\mathbb{R}^n$.

Each real number x_i is called a **component**, or **element**, of the vector. The **size** of a vector is the number of components it contains. Two vectors x and y are said to be **equal**, $x = y$, if they have the same size, and $x_i = y_i$ for all i.

Example 3.1 (functions):

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^3$$

is a vector in $\mathbb{R}^3$, and represents a single point in $\mathbb{R}^3$.

Remark. Vectors can be much more general than the popular $x \in \mathbb{R}^n$. For example,

(a) $x \in \mathbb{C}^n$**:** Here, x_i is a complex number

(b) $x \in \mathbb{R}^{m \times n}$: Here, x is an $m \times n$ matrix with real entries:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$$

(c) Functions are also vectors!

   (a) For example, x could be a continuous function over the real line x.

   (b) We usually write $x(t)$ as "x as a function of t", where $t \in \mathbb{R}$ and $x(t_1), x(t_2)$, etc. are components of the vector:

$$x = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_n) \end{bmatrix}$$

Definition 3.2 (state): The minimum information you need to completely characterize a system at a given point in time, without any need for more information about the past of the system.

Example 3.2 (quadrotor): The 3D position, angle, velocity, and angular velocity of a quadrotor at a particular time can be represented as - you guessed it - a vector $X \in \mathbb{R}^{12}$. The dot above the components

$$X = \begin{bmatrix} x \\ y \\ z \\ \text{roll} \\ \text{pitch} \\ \text{yaw} \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\text{roll}} \\ \dot{\text{pitch}} \\ \dot{\text{yaw}} \end{bmatrix}$$
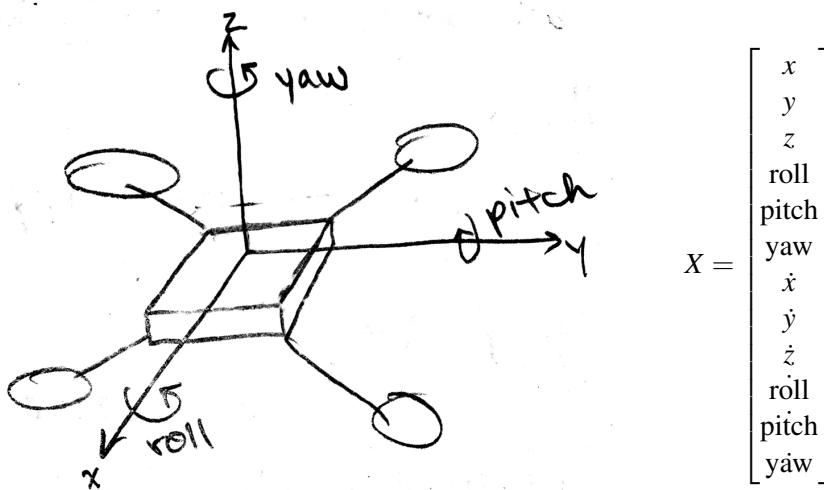
Figure 1: A quadrotor like the ones Claire studies in her lab, and the vector containing the information needed to determine its state.

represents the derivative with respect to time.

Example 3.3 (color): This is relevant to the lab we will do!

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x \in \mathbb{R}^3$$

can represent a color, with its components giving the red, green, and blue (RGB) intensity values. Typically x will fall in a range that reflects the range of the color (think 0 - 255).

Example 3.4 (position): $x \in \mathbb{R}^N$ can represent the sample values of a quantity at N time points. Think of a car moving along a line. The positions at $t_1, t_2, \cdots, t_N$ can be represented in a vector:

$$X = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{bmatrix}$$

Here, $X \in \mathbb{R}^N$ is the vector and $x(t_i)$ represents the position at time t_i.

Example 3.5 (image): A black and white image of $M \times N$ pixels can be represented by a vector of length MN, with the components giving greyscale levels at the pixel location.
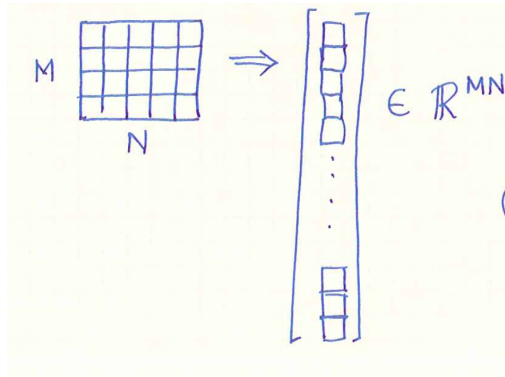
What about a color image? The image could be stored in a vector in $\mathbb{R}^{3MN}$, because each of the MN pixels has values for red, green, and blue.

What about a color video? This could be stored in a vector in $\mathbb{R}^{MN(\text{number of frames in the video})}$.

Special Vectors

Definition 3.3 (zero vector): A **zero vector** is a vector with all the components equal to zero, usually just represented as 0. You can tell the size of the vector from the context: if $\mathbf{x} \in \mathbb{R}^n$ is added to 0, then 0 must also be $\in \mathbb{R}^n$.

Definition 3.4 (unit vector): A **unit vector** is a vector with all components equal to 0 except for one element, which is equal to 1. A unit vector where the ith position is equal to 1 is written as e_i. We can denote the 3 unit vectors in $\mathbb{R}^3$ as

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Vector Addition

Two vectors of the same size and in the same space (e.g. complex numbers, reals, etc.) can be added together by adding their corresponding components:

$$\begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1+2 \\ 3.5-1 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \\ 3 \end{bmatrix}$$
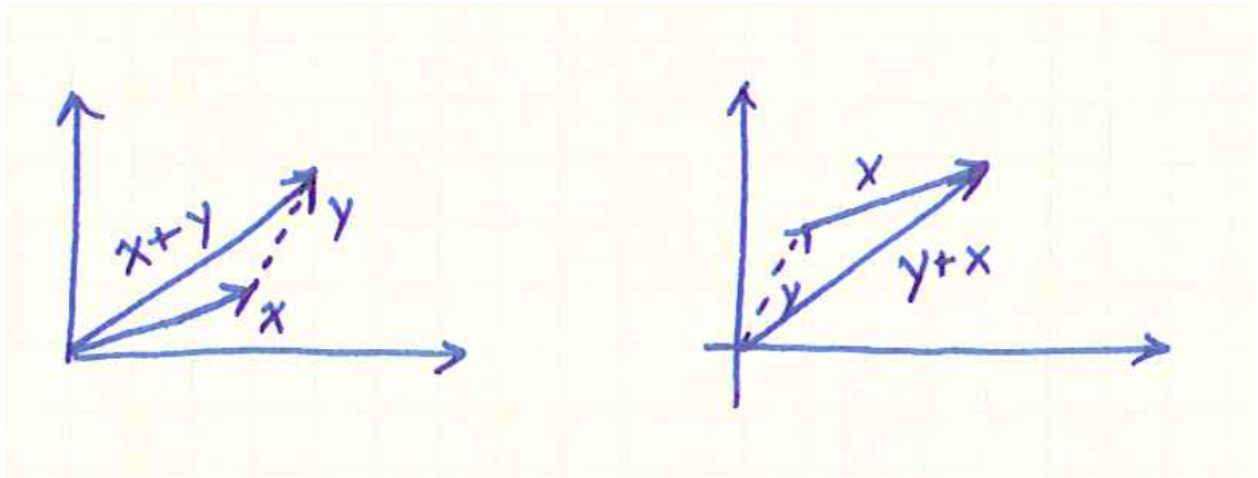
Visually, you place the first vector's tail at the origin, and attach the second vector's tail to the first vector's head. The resulting vector from the origin to the second vector's head is the result vector.

Properties of vector addition: for $x, y, z \in \mathbb{R}^n$

|	
commutative	$x + y = y + x$
associative	$(x + y) + z = x + (y + z)$
zero	$x + 0 = x$
additive inverse	$x + (-x) = 0$

Scalar Multiplication

We can multiply a vector by a number, called a **scalar**, that is in the same space as the vector (e.g. also in the reals or in the complex numbers). Just multiply each of the components of the vector by the scalar:

$$(-3) \begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -10.5 \\ 0 \end{bmatrix}$$

In general, this looks like

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$
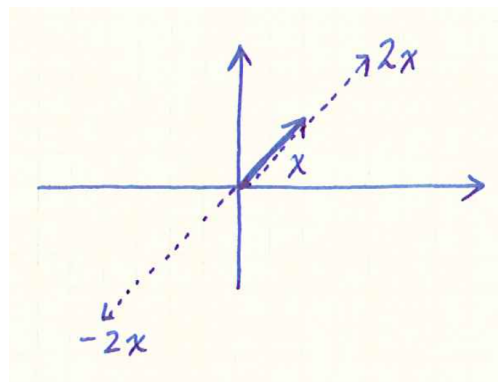
Example 3.6 (negative vector): $-x$, where $x \in \mathbb{R}^n$, is just $(-1)x$ (-1 is a scalar).

Example 3.7 (zero vector): $0x = 0$

Properties of scalar multiplication

associative	$(\alpha\beta)x = \alpha(\beta x)$
distributive	$(\alpha + \beta)x = \alpha x + \beta x$
identity	$1x = x$

Functions as vectors

Functions reflect many of the same properties and operations that vectors have.

Vector spaces	**Functions**
vector addition	$f(x) + g(x)$
scalar multiplication	$\alpha f(x)$
zero vector	x-axis

Inner Products

Definition 3.5 (inner product): The inner product (or dot product) between two vectors $x, y \in \mathbb{R}^n$ is

$$x^T y = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbb{R}^n$$

$$= \sum_{i=1}^{n} x_i y_i$$

We also denote the dot product as $x \cdot y$ or $< x, y >$.