Lecture notes by Dasol Yoon (02/03/2015)

Imaging

Examples of imaging:

- Get readings from the brain, then construct an image from the information
- Milk, Juice, and Empty bottles arranged in a 3x3 carton.
  - Example carton:

    (See Figure 1)
  - If we shine light through the carton, we can try to figure out what types of bottles are in that row or column. A Milk bottle adds 3 units, a Juice bottle adds 2 units, and an Empty bottle adds 1 unit. So in our example carton, we would have 5 units total for the first row.

    (See Figure 2)
  - We have a closed carton, and we want to figure out what types of bottles are inside. A way that we can do this is by shining light on each row, column, and diagonal.

    (See Figure 3)
  - To solve this, we just combine these equations:

$$rows: \begin{cases} x_{1,1}+x_{1,2}+x_{1,3}=5 \\ x_{2,1}+x_{2,2}+x_{2,3}=6 \\ x_{3,1}+x_{3,2}+x_{3,3}=4 \end{cases} \quad columns: \begin{cases} x_{1,1}+x_{2,1}+x_{3,1}=6 \\ x_{1,2}+x_{2,2}+x_{3,2}=3 \\ x_{1,3}+x_{2,3}+x_{3,3}=6 \end{cases} \quad diagonals: \begin{cases} x_{1,1}+x_{2,2}+x_{3,3}=5 \\ x_{1,3}+x_{2,2}+x_{3,1}=5 \end{cases}$$

to get the result.

(See Figure 4)

We have a 9x9 matrix that we want to image. We can take the following steps:

- Shine light onto the image
- Collect reflected light
- Collect measurement that says whether the pixel is black or white

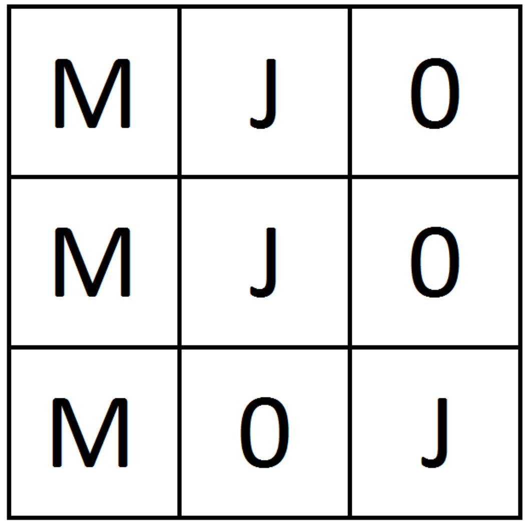

Figure 1: An example carton.

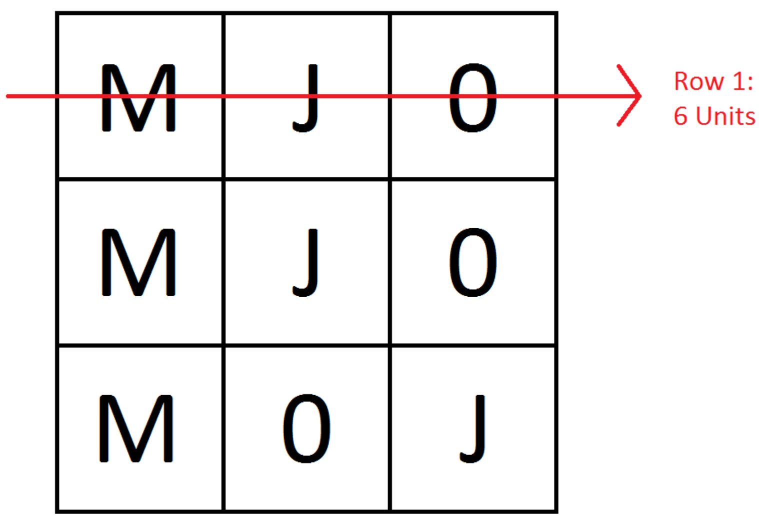

Figure 2: Shining a light through row 1 of the example carton.

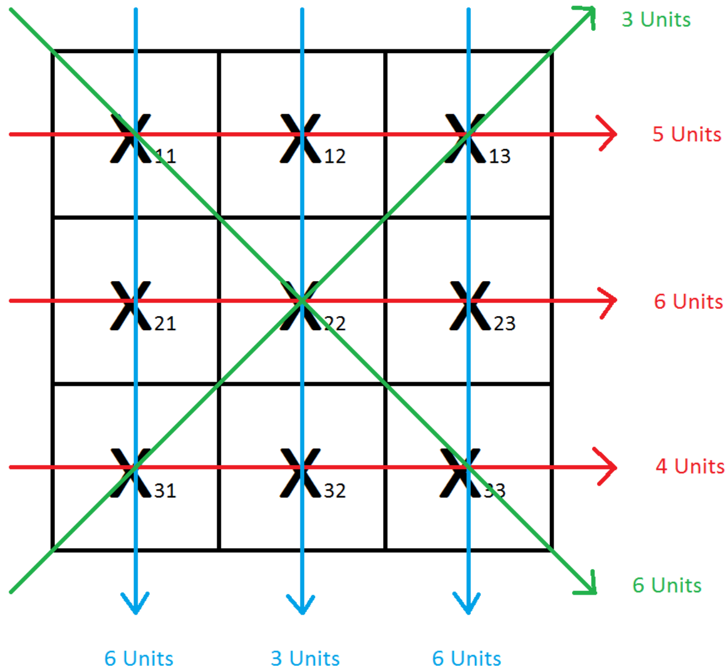

Figure 3: Shining lights through our mystery carton.

M	O	O
J	O	M
O	O	J

Figure 4: Our solved carton.

- Ideally, we would shine light on every pixel:

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9 \end{pmatrix} = \begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9 \end{pmatrix}$$

  This is like the action of taking a picture and representing it with the matrix multiplication

- What if the photodetector had more constraints?

$$\begin{pmatrix} 1&1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9 \end{pmatrix} = \begin{pmatrix} x_1+x_2\\x_2+x_3\\x_3+x_4\\x_4+x_5\\x_5+x_6\\x_6+x_7\\x_7+x_8\\x_8+x_9\\x_9 \end{pmatrix}$$

  We do not want to multiply by an 8x9 matrix because we could be losing information!

Linear Equations

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a1,3\\ a_{2,1} & a_{2,2} & a2,3\\ a_{3,1} & a_{3,2} & a3,3 \end{pmatrix} \begin{pmatrix} x_1\\x_2\\x_3 \end{pmatrix} = \begin{pmatrix} b_1\\b_2\\b_3 \end{pmatrix} \quad We\ want\ to\ try\ to\ get\ an\ invertible\ matrix.$$

$$a_{1,1}x_1+a_{1,2}x_2+a_{1,3}x_3=b_1 \quad a_{2,1}x_1+a_{2,2}x_2+a_{2,3}x_3=b_2 \quad a_{3,1}x_1+a_{3,2}x_2+a_{3,3}x_3=b_3 \tag{1}$$

$$\begin{pmatrix} 0.5&0.5&0&0\\ 0&0.5&0.5&0\\ 0&0&0.5&0.5\\ 0&0&0&0.5 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix} = \begin{pmatrix} 0.5\\0.5\\0.5\\0.5 \end{pmatrix} \quad This\ would\ produce\ an\ all-gray\ bar.$$

$$\begin{pmatrix} 0.9&0.1&0&0\\ 0&0.9&0.1&0\\ 0&0&0.9&0.1\\ 0&0&0&0.9 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix} = \begin{pmatrix} 0.1\\0.9\\0.1\\0.9 \end{pmatrix} \quad This\ would\ be\ closer\ to\ a\ gradient.$$

Vector Spaces

A **Basis** is the minimum number of representations of the vectors in a vector space.
A **vector space** $(\mathbb{V}, \mathbb{F})$ is a set of vectors $\mathbb{V}$, a set of scalars $\mathbb{F}$, and two operators:

- Vector Addition
  - Associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
  - Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - Additive Identity: $\vec{0}$ such that $\vec{v} + \vec{0} = \vec{v}$
  - Additive Inverse: $\vec{v}$, $\vec{w}$ such that $\vec{v} + \vec{w} = 0$
- Scalar Multiplication
  - Associative: $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$
  - Commutative: $(\lambda \cdot \vec{u}) \cdot \vec{v} = \lambda \cdot (\vec{u} \cdot \vec{v})$
  - Multiplicative Identity: $1 \cdot \vec{v} = \vec{v}$
  - Zero: $\vec{0} \cdot \vec{v} = \vec{0}$

Distributive Law: Scalar multiplication will distribute across vector addition.