Lecture notes by Andy Wang (02/10/2015)

7.1 Row Operations

The process of row operations allows us to solve matrix equations such as $Ax = b$.

7.1.1 Example 1

Consider the following system of two equations.

x + y = 3
2x + y = 4

We can rewrite this system as a matrix equation.

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Of course, we could attempt to solve this matrix equation by creating an augmented matrix and row reducing until we ahve values for x and y; however, it would be much easier if we developed a process where we could single out variables like x and y. This process is known as *Gaussian Elimination*. Fun Fact: Gaussian Elimination was actually discovered 2000 years before Gauss in China.

Let us first give names to the rows of our matrices, R_1 to the top row and R_2 to the bottom row.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

In order to single out x and y, we want the terms on the diagonal of the 2 by 2 matrix to be 1, and all other terms to be 0. By turning this matrix into the identity matrix, we can single out our variables during matrix multiplication. To accomplish this, we will row reduce our matrices with row operations beginning with: $R_2 \to R_2 - 2R_1$.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

This process may seem familiar to you. It is simply a fancier way to perform algebraic processes (i.e. $(2x+y) - 2(x+y) = -2$ is the same as above). Now, we still want the bottom right term in the matrix to be 1, so we will factor out a -1 from the bottom row: $R_2 \to (-1)R_2$.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Finally, we will row reduce the first row: $R_1 \rightarrow R_1 + (-1)R_2$.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now if we were to carry out matrix multiplication for this matrix equation, we could conclude that,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

7.1.2 Final Focus

To sum it all up, we performed three kinds of row operations in Example 1. They are as follows:

Row Operations	2 x 2 Example
1. Row Replacement	$R_1 \rightarrow R_1 + \alpha R_2$
2. Row Scaling	$R_1 \rightarrow \alpha R_1$
3. Row Exchange	$R_1 \leftrightarrow R_2$

Finally, we can sum up our process by saying that we started out with an arbitrary n by n matrix. We then row reduced it to an n by n triangular matrix (all the terms below the main diagonal are 0). Finally, we row reduced it to a matrix where the only non-zero terms are along the main diagonal.

7.2 Matrix Inversion

We will first define the definition of an inverse matrix. An inverse matrix satisfies the following condition for an arbitrary invertible matrix, A. $AA^{-1} = A^{-1}A = I$ where I is the identity matrix. To represent this with the actual terms of a matrix, let A equal to the 2 by 2 matrix that we used in example 1 above.

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let b_{ij} represent the terms of the inverse matrix A^{-1} where i is the value of the row of b, and j is the value of the column. We can also further represent the equation above as two more familiar looking equations by breaking up the inverse matrix A^{-1} and the identity matrix I into their column vectors.

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad AND \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So now the important question is how do we figure out what the terms of an inverse matrix actually are? Using the matrices above, we will form an augmented matrix by stacking the two next to each other to form a 2 by 4 matrix.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array} \right]$$

Now, performing the same row reduction techniques as in example one, our goal is to turn the 2 by 2 matrix to the left side of the dashed line into the identity matrix, which is currently to the right side of the dashed line.

Step 1: $R_2 \rightarrow R_2 - 2R_1$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

Step 2: $R_2 \rightarrow (-1)R_2$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Step 3: $R_1 \rightarrow R_1 + (-1)R_2$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Ultimately, we can conclude that the inverse matrix is as follows:

$$A^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

As an exercise, you can check this by proving that $AA^{-1} = A^{-1}A = I$.