Lecture notes by Christine Wang (03/31/2015)

Solving underdetermined sets of equations

So far, we've learned how to solve overdetermined systems of equations (where there are more equations than variables) using least-squares. In this lecture, we solve under-determined system of equations using a slightly different approach.

Suppose $A\vec{x} = b$, where A is an $n \times m$ matrix, b is an unknown n-vector, and x is an unknown m-vector. Assume $n < m$ — there are fewer constraints than unknowns.

$$\begin{bmatrix} & & A & & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

We also assume that A has full row rank, rank(A) = n. (Rank is the number of independent columns.)

Example 1: How do you represent the line $x_1 + x_2 = 1$ as $A\vec{x} = b$?

We can formulate this as an underdetermined set of equations.

$$A\vec{x} = b$$

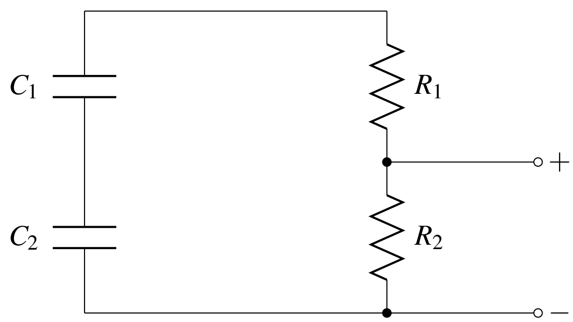
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

In general, in this case, $A\vec{x} = b$ has an infinite number of solutions. We can pick one of these solutions by finding the one with the minimum norm.

$$\min_{\vec{x}} ||\vec{x}||^2 \text{ such that } A\vec{x} = b$$

In this case, it turns out to be the point $(\frac{1}{2}, \frac{1}{2})$.

Example 2: Suppose C_1, C_2, R_1, R_2 are the components given to you, and they are organized in the setting below. You would like to have voltage b across resistor R_2 in the figure. We would like to charge the capacitors to voltages x_1 and x_2 to get b. Find the capacitor voltages x_1 and x_2 to minimize $||x||^2$. b **is the voltage between the 2 unconnected terminals to the right of the circuit.**



Solution: Notice here that what matters to determine b is just the total volatage $x_1 + x_2$. So we only really have one constraint, but again we have two variables that we can choose. We know from the equation for a voltage divider that if the voltages on C_1 and C_2 are x_1 and x_2 then

$$b = (x_1 + x_2)\frac{R_2}{R_1 + R_2}$$

Now, we can set:

$$A = \begin{bmatrix} \frac{R_2}{R_1+R_2} \\ \frac{R_2}{R_1+R_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{R_2}{R_1+R_2} \\ \frac{R_2}{R_1+R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b = (x_1 + x_2)\frac{R_2}{R_1 + R_2}$$

in the desired format $A\vec{x} = b$.

We solve this constrained optimization problem using the method of Lagrange multipliers, where we add a term to the quantity to be minimized. The vector $\vec{\lambda}$ is the vector of the Lagrange multipliers.

$$\min_{\vec{x},\vec{\lambda}} \|\vec{x}\|^2 + \vec{\lambda}^T(b - A\vec{x}) \tag{1}$$

Differentiating with respect to $\vec{x}$ and setting the result to 0 gives

$$\frac{\partial}{\partial \vec{x}}(\vec{x}^T\vec{x} + \vec{\lambda}^T(b - A\vec{x})) = 0$$
$$2\vec{x}^T - \vec{\lambda}^T A = 0$$
$$2\vec{x} - A^T\vec{\lambda} = 0$$

Left-multiplying by A:

$$2A\vec{x} - AA^T\vec{\lambda} = 0$$
$$\therefore \vec{\lambda} = (AA^T)^{-1}2A\vec{x}$$

Differentiating (1) with respect to $\vec{\lambda}$ and setting the result to zero:

$$A\vec{x} = b$$
$$\vec{\lambda} = (AA^T)^{-1}2b$$

Since $2\vec{x} - A^T\vec{\lambda} = 0$,

$$\vec{x} = A^T(AA^T)^{-1}b$$

This is the least-norm solution to $A\vec{x} = b$.