EECS 16A Designing Information Devices and Systems I
Spring 2015

Note 20

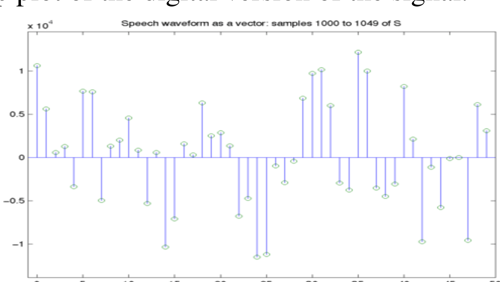
Lecture notes by Ankit Mathur (4/7/15)

Signals

We're going to be talking about signals and systems that process those signals. At its heart, a signal is just a function, and we try to convey information by mapping it to a given signal. Signals therefore have a domain and a range. There are several different types of signals, but the ones we plan on studying are –

1. Continuous time signals (analog signals) - signals that you can draw across the real axis. Usually, the variable in the x direction is time (we have $x(t)$, but it does not have to be t that is the variable). A continuous time signal is defined mathematically by the domain and range where $x : \mathbb{R} \mapsto \mathbb{R}$ (or $\mathbb{C}$).
The following is an example of a speech waveform represented as an analog signal over time:

2. Discrete time signals - signals that are only defined on discrete values in the domain (they are undefined at all other points). This signal is not a function of a continuous argument, so it can map over a continuous space by sampling across it. Because these signals are not defined across all values of x, they cannot be drawn as a continuous graph. Mathematically, the domain and range that define these are $x : \mathbb{Z} \mapsto \mathbb{R}$ (or $\mathbb{C}$).
The following image is the speech waveform from before, except sampled and represented as a lollipop plot of the digital version of the signal.



There are 2 views, or broad domains, that are going to be relevant for signal analysis to us: the Time Domain view (signal's footprint in the context of time) and the Frequency Domain View (decomposing such signals).

Transmitting a Signal on a Carrier

In the previous lecture, we were going over an example where there were two pieces of information on a single function, and we wanted to try to extract each of these signals. We had the following equation –

$$x(t) = X_1 cos(\omega_1 t) + X_2 cos(\omega_2 t) \tag{1}$$

where X_1 and X_2 are the two different pieces of information and $cos(\omega_1 t)$ and $cos(\omega_2 t)$ are the two carrier signals for each of the frequencies ω_1 and ω_2.
There are actually 2 ways to transmit information

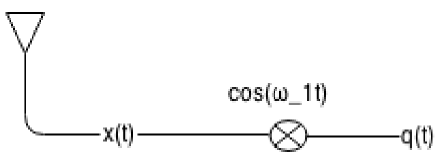
a. Time-division-multiplexing - the idea that people must take turns talking so only 1 person can be talking at a time (e.g. the dinner table).

b. Frequency-division-multiplexing - the idea that several people can talk at the same time because the signal contains the information from all of them at once.

In the example that we will be working on, we are working with a case of frequency-division multiplexing. We are, in this case, working with a special case where X_1 and X_2 are constants. The more general form of the problem where, instead of constants, X_1 and X_2 are actually functions! The equation, which can carry arbitrarily large numbers of signals (though with diminishing returns), is

$$x(t) = X_1(t) cos(\omega_1 t) + X_2(t) cos(\omega_2 t)$$

Going back to the original problem, we used the following setup to generate a new function $q(t)$ from $x(t)$ to help us recover the original function.



We are essentially multiplying $x(t)$ by $cos(\omega_1 t)$. That means we get that

$$q(t) = X_1 cos^2(\omega_1 t) + X_2 cos(\omega_1 t) cos(\omega_2 t) \tag{2}$$

We can use some trigonometric identities to transform this equation into something usable. We know that –

$$cos^2(x) = \frac{1}{2} + \frac{1}{2} cos(2x) \tag{3}$$

$$cos(\alpha) cos(\beta) = \frac{1}{2}(cos(\alpha + \beta) + cos(\alpha - \beta)) \tag{4}$$

We use equation 3 on the first part of $q(t)$ and equation 4 on the second part of $q(t)$.
From that, we get –

$$q(t) = \frac{X_1}{2} + \frac{X_1}{2} cos(2\omega_1 t) + \frac{X_2}{2} cos((\omega_1 + \omega_2)t) + \frac{X_2}{2} cos((\omega_1 - \omega_2)t)$$

To avoid some clutter (we don't have to make this assumption, but it simplifies the explanation), let us assume that $\omega_1 < \omega_2$. Also, we assume without loss of generality that $\omega_1 = k_1\omega_0$ and $\omega_2 = k_2\omega_0$ where ω_0 is some fundamental frequency and $k_1, k_2 \in \mathbb{N}$.
From this, we know that $cos(\omega_0 t)$ has the period $T_0 = \frac{2\pi}{\omega_0}$. However, because of the assumptions we made about how ω_1 and ω_2 relate to ω_0, We know that $cos(\omega_1 t)$ has the period $T_1 = \frac{T_0}{k_1}$, and that $cos(\omega_2 t)$ has the period $T_2 = \frac{T_0}{k_2}$. Now, we heard from the last lecture that we wanted to average $q(t)$ over some time interval so that the terms from the cosine graphs go to 0 because of the periodic nature of the cosine wave. Therefore, we need a way to average this continuous function. The following equation from basic calculus lets us do this –

$$\bar{x} = \frac{1}{b-a}\int_a^b x(t)dt$$

To understand this, we know that the discrete analog for this average value of $x_0, x_1, x_2 \dots x_{N-1}$ should be $\bar{x} = \frac{1}{N}\sum_{k=0}^{N-1} x_k$ We now want to average this function over some interval that matches the cycles of both frequencies, and the most obvious interval to average over in this case would be to average over T_0. Thus, we get –

$$\bar{q} = \frac{X_1}{2} + \frac{X_1}{2}\frac{1}{T_0}\int_0^{T_0} cos(2\omega_1 t) + \frac{X_2}{2}\frac{1}{T_0}\int_0^{T_0} cos((\omega_1+\omega_2)t) + \frac{X_2}{2}\frac{1}{T_0}\int_0^{T_0} cos((\omega_1-\omega_2)t) \tag{5}$$

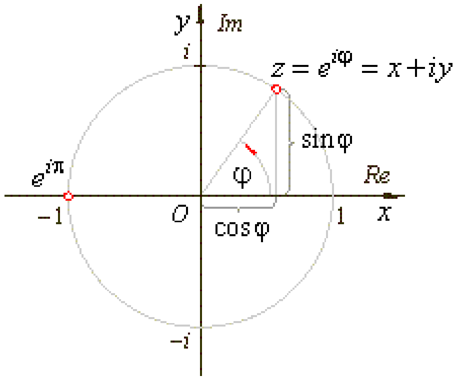
For the first integral term, we average over $2k_1$ cycles ($\frac{T_1}{2} = \frac{T_0}{2k_1} \implies 2k_1$ cycles). For the second term, $k_1 + k_2$ cycles, and, for the third term, $k_2 - k_1$ cycles, where the number of cycles is found by the same process as the first integral term.
The same strategy will work in the general case described earlier.
All of this really works because $\int_0^T cos(\omega_1 t)cos(\omega_2 t)dt = 0$ if $\omega_1 \neq \omega_2$, and this works for any interval over T (from $-\frac{T}{2}$ to $\frac{T}{2}$ or T to $2T$). This is, in general denoted as $\int_{<T>}$ (an integral over a continuous interval of duration T). This is actually an inner product, and these signals will end up being orthogonal.

Phasors

A phasor is a complex number representing a sinusoidal function whose amplitude, frequency, and phase are time-invariant.
We're going to describe the phasor denoted by $q(t) = e^{it}$
The spectrum of a signal is its decomposition in terms of different complex exponentials, each of which represents some different frequency. If you can get signal into a linear composition of such signals, you have decomposed it into its constituent frequencies. From these you can see how much of a signal is in some complex signal. An example of this in the real world is the equalizer for music, which adjusts how much of the constituent parts of a song are output. For example, in the case of $q(t)$,

$$x(t) = X_0 e^{i\omega_0 t} + X_1 e^{i\omega_1 t} + \cdots + X_{N-1} e^{i\omega_{N-1} t} \tag{6}$$

Now, consider the complex plane and say you have some particle whose position is described by $q(t)$



(Note that in this image, we see ω. In our math, we will replace this with t in all instances where we see ω here.)

We want to prove the claim that $q(t)$ is always on the unit circle. We know that $q(0) = 1$. Because we have the complex plane, we know that

$$e^{it} = cos(t) + isin(t)$$

We want to assert that complex exponentials will have the same properties as the function $r = e^{\alpha t}$, which has the critical property that $\dot{r}(t) = \frac{\partial r}{\partial t} = \alpha e^{t} = \alpha r(t)$

We now work with $q(t)$.

$$\begin{aligned} q(t) &= e^{it} \\ q(0) &= 1 \\ \dot{q}(t) = ie^{it} &= iq(t) \text{ where } i = \sqrt{-1} \end{aligned}$$

Now, for any complex number $z = a + ib$, $iz = -b + ia$, and iz is orthogonal to z.

Then, we can say that for $q(t)$,

$$\begin{aligned} q(t) &= a(t) + ib(t) \\ \dot{q}(t) = -b(t) + ia(t) &= \dot{a}(t) + i\dot{b}(t) \\ \implies \dot{a}(t) &= -b(t) \\ \implies \dot{b}(t) &= a(t) \end{aligned}$$

We get the last two implications just by comparing these two different real and complex parts of $\dot{q}$. Note that since $\dot{q}(t) = ie^{it}$, $\dot{q}(t) = 0$. Therefore, we can see that $\dot{q}(t)$ is always perpendicular to $q(t)$, which, intuitively means that $q(t)$ should go in a circle (see uniform circular motion from physics), so we know that is one trajectory that seems like the right answer. We want to prove this is the only trajectory possible for $q(t)$.

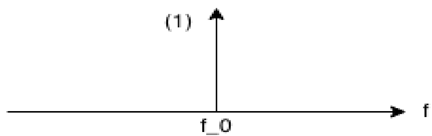
Start by multiplying the two equations we got from the implication

$$\begin{aligned} a(t)\dot{a}(t) &= -b(t)\dot{b}(t) \\ a(t)\dot{a}(t) + b(t)\dot{b}(t) &= 0 \\ 2a(t)\dot{a}(t) + 2b(t)\dot{b}(t) &= 0 \end{aligned}$$

By integrating, and remembering the chain rule, and the initial condition that $q(t) = 1$, which means that $a(t) = 1$ and $b(t) = 0$.

$$
\begin{aligned}
a^2(t) + b^2(t) &= C \\
1^2 + 0^2 = 1 &= C \\
a^2(t) + b^2(t) &= 1
\end{aligned}
$$

Thus, we have proven that the trajectory is in the path of a unit circle, since this equation matches the equation of a unit circle.
We can also note that $q(t) = e^{it}$ takes 2π seconds to complete 1 revolution. That implies that $q(t) = e^{i2\pi t}$ takes 1 second to complete a full cycle and is a signal of frequency 1 Hz.
More generally, $e^{i2\pi f_0 t}$ takes f_0 cycles per second. This is a single frequency signal, which means it can be expressed by a single complex exponential.
In the frequency domain, its representation, represented by $Q(f)$ is



where f_0 is the cycles per second measure and the number next to the arrow denotes its "strength". This is how you represent the spectrum of a signal.
Another way to express this is that $q(t) = e^{i\omega_0 t}$ and $\omega_0 = 2\pi f_0$.
Now, if you wanted to draw the $x(t) = cos(2\pi f_0 t)$ and the following graph represents the spectrum of this signal.

We can actually construct the cosine signal from the e^{it} signal we have been analyzing.

$$
\begin{aligned}
e^{it} &= cos(t) + isin(t) \\
e^{i2\pi f_0 t} &= cos(2\pi f_0 t) + isin(2\pi f_0 t) \\
e^{-i2\pi f_0 t} &= cos(2\pi f_0 t) - isin(2\pi f_0 t)
\end{aligned}
$$

From these last 2 expressions we can just add these expressions and divide by 2 to get to cosine. This leaves us with

$$x(t) = cos(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2}$$

Therefore, the cosine signal is actually complex and is not a single frequency signal.