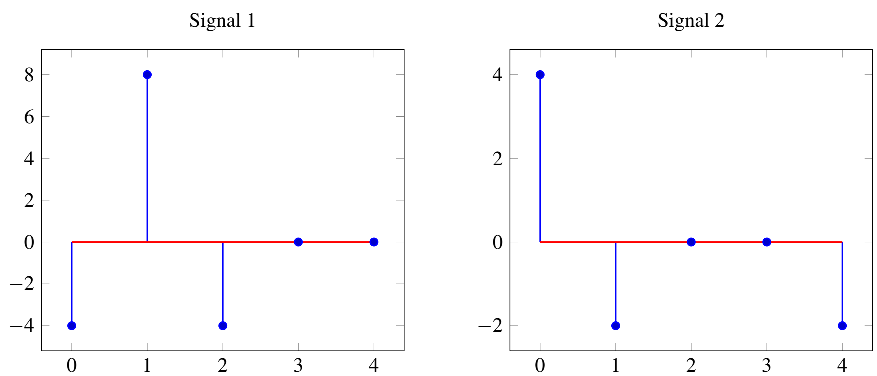
EECS 16A Designing Information Devices and Systems I
Spring 2018 Discussion 11B

## 1. Correlation

You are given the following two signals:

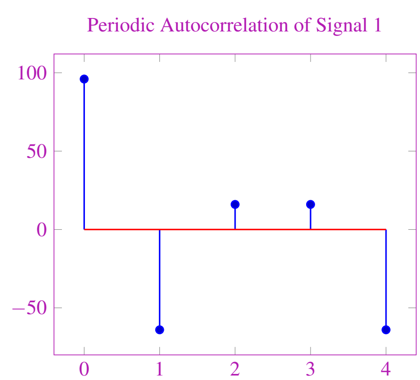



Assume that both signals are periodic with period 5, that is, each plot shows one full period of a periodic signal.

(a) Sketch the autocorrelation (correlation with itself) of signal 1.

**Answer:**

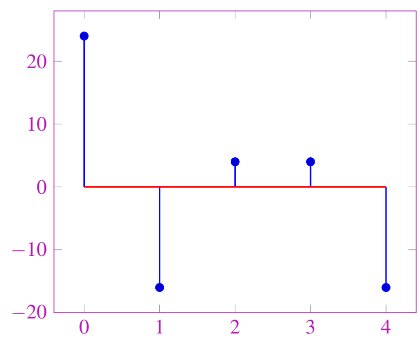
The autocorrelation is as follows. Autocorrelation is a special case of cross-correlation (it is the cross-correlation of a signal with itself). See the answer for part (c) for an example of how to compute cross-correlation.




(b) Sketch the autocorrelation of signal 2.

**Answer:**

Periodic Autocorrelation of Signal 2




(c) Sketch the cross-correlation of signal 1 with signal 2. Suppose we know that signal 2 is a delayed (and attenuated) version of signal 1. What does the cross-correlation tell us about the delay?

**Answer:**

Represent signal 1 as the vector $\vec{x} = \begin{bmatrix}-4 & 8 & -4 & 0 & 0\end{bmatrix}^T$. We write $\vec{x}[0] = -4, \vec{x}[1] = 8$, etc. Similarly, represent signal 2 as the vector $\vec{y} = \begin{bmatrix}4 & -2 & 0 & 0 & -2\end{bmatrix}^T$.

To compute the cross-correlation $\mathbf{C}_{\vec{y}}^T\vec{x}$, we circularly shift the vector $\vec{y}$ and compute the inner product of the shifted $\vec{y}$ with the original $\vec{x}$. For example, to compute the cross-correlation at 0 (corresponding to shifting $\vec{y}$ by 0):

$$\mathbf{C}_{\vec{y}}^T\vec{x}[0] = \langle \vec{x}, \vec{y} \rangle = (-4)(4) + (8)(-2) + (-4)(0) + (0)(0) + (0)(-2) = -32$$

To compute the cross-correlation at lag 1, we first circularly-shift $\vec{y}$ right by 1. Call this shifted version $\vec{y}_1$.

$$\vec{y} = \begin{bmatrix}4 & -2 & 0 & 0 & -2\end{bmatrix}^T$$
$$\vec{y}_1 = \begin{bmatrix}-2 & 4 & -2 & 0 & 0\end{bmatrix}^T$$

Then:

$$\mathbf{C}_{\vec{y}}^T\vec{x}[1] = \langle \vec{x}, \vec{y}_1 \rangle = 48$$

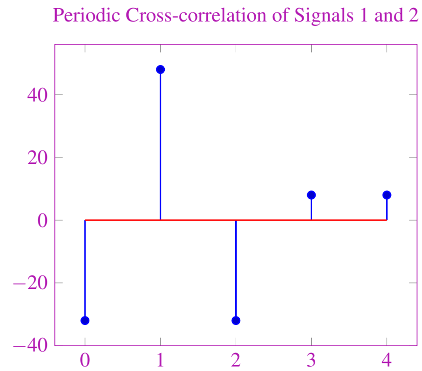
It can be useful to visualize correlations as "shifting $\vec{y}$ along $\vec{x}$" and pointwise-multiplying the two signals. For example, to compute $\mathbf{C}_{\vec{y}}^T\vec{x}[1]$ as above:

| $\vec{x}$ | $-4$ | | $8$ | | $-4$ | | $0$ | | $0$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{y}_1$ | $-2$ | | $4$ | | $-2$ | | $0$ | | $0$ | |
| $\langle \vec{x}, \vec{y}_1 \rangle$ | $8$ | $+$ | $32$ | $+$ | $8$ | $+$ | $0$ | $+$ | $0$ | $= 48$ |

To compute $\mathbf{C}_{\vec{y}}^T\vec{x}[2]$, we shift $\vec{y}$ by 2:

| $\vec{x}$ | $-4$ | | $8$ | | $-4$ | | $0$ | | $0$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{y}_2$ | $0$ | | $-2$ | | $4$ | | $-2$ | | $0$ | |
| $\langle \vec{x}, \vec{y}_2 \rangle$ | $0$ | $+$ | $-16$ | $+$ | $-16$ | $+$ | $0$ | $+$ | $0$ | $= -32$ |

Continuing in this way, we find $\mathbf{C}_{\vec{y}}^T\vec{x} = \begin{bmatrix}-32 & 48 & -32 & 8 & 8\end{bmatrix}^T$, as shown below.




The peak of this cross-correlation is at lag $m = 1$, meaning that the best alignment of signal 2 against signal 1 is when signal 2 is shifted right by $m = 1$.

**Remark (optional):** There is a subtlety here: it appears that signal 2 actually *leads* signal 1. That is, signal 2 appears "delayed" by $(-1)$ time steps. However, recall that the signals are periodic with period 5. Thus, the actual delay could have been $-1+5=4$, or $-1+5\cdot 2=9$, or in general $-1+5k$ for some integer $k$.

**2. Autocorrelation Peak**

Let $\mathbf{C}_{\vec{x}}^T\vec{x}$ be the autocorrelation of an $N$-periodic signal $\vec{x}$. Prove that $\mathbf{C}_{\vec{x}}^T\vec{x}[0] \geq \left|\mathbf{C}_{\vec{x}}^T\vec{x}[m]\right| \; \forall m$. In other words, the autocorrelation peak (maximum value of autocorrelation) of any periodic signal always occurs at lag $m = 0$.

**Answer:**

**Method 1:** First, we define $\vec{x} = \begin{bmatrix} \vec{x}[0] \\ \vec{x}[1] \\ \cdots \\ \vec{x}[N-1] \end{bmatrix}$ and $\vec{x}_m = \begin{bmatrix} \vec{x}[(-m)_N] \\ \vec{x}[(-m+1)_N] \\ \cdots \\ \vec{x}[(-m+N-1)_N)] \end{bmatrix}$. Recall the geometric interpretation of the inner product $\langle \vec{x}, \vec{y} \rangle = ||\vec{x}|| \, ||\vec{y}|| \cos\theta$. Applying this, we find that

$$\begin{aligned} \mathbf{C}_{\vec{x}}^T\vec{x}[0] &= \langle \vec{x}, \vec{x} \rangle \\ &= ||\vec{x}|| \, ||\vec{x}|| \cos(0) \\ &= ||\vec{x}||^2 \\ \left|\mathbf{C}_{\vec{x}}^T\vec{x}[m]\right| &= |||\vec{x}|| \, ||\vec{x}_m|| \cos\theta| \\ &= ||\vec{x}||^2 |\cos\theta| \\ &\leq ||\vec{x}||^2 = \mathbf{C}_{\vec{x}}^T\vec{x}[0], \end{aligned}$$

where the inequality comes from the fact that $\cos\theta$ is bounded between $-1$ and $1$.

**Method 2:** We make note of the expansion

$$(\vec{x}[n+m] - \vec{x}[n])^2 = \vec{x}[n+m]^2 + \vec{x}[n]^2 - 2\vec{x}[n+m]\vec{x}[n]$$

and compute

$$\begin{aligned}
\mathbf{C}_{\vec{x}}^{T}\vec{x}[m] &= \sum_{n=0}^{N-1} \vec{x}[n+m]\vec{x}[n] \\
&= \sum_{n=0}^{N-1} \frac{1}{2}\left(\vec{x}[n]^2 + \vec{x}[n+m]^2 - (\vec{x}[n+m]-\vec{x}[n])^2\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{N-1}\vec{x}[n]^2 + \sum_{n=0}^{N-1}\vec{x}[n+m]^2 - \sum_{n=0}^{N-1}(\vec{x}[n+m]-\vec{x}[n])^2\right) \\
&= \frac{1}{2}\left(\mathbf{C}_{\vec{x}}^{T}\vec{x}[0] + \mathbf{C}_{\vec{x}}^{T}\vec{x}[0]\right) - \frac{1}{2}\sum_{n=0}^{N-1}(\vec{x}[n+m]-\vec{x}[n])^2 \\
&= \mathbf{C}_{\vec{x}}^{T}\vec{x}[0] - \frac{1}{2}\sum_{n=0}^{N-1}(\vec{x}[n+m]-\vec{x}[n])^2 \\
&\leq \mathbf{C}_{\vec{x}}^{T}\vec{x}[0],
\end{aligned}$$

where the inequality comes from the fact that we are subtracting a sum of squares, which must be non-negative.

**Method 3:** Imagine that $\vec{x}$ is just a list of numbers. Now imagine that you are allowed to rearrange the numbers in $\vec{x}$ to form a new vector $\tilde{\vec{x}}$. How would you rearrange the numbers to maximize $\langle \vec{x}, \tilde{\vec{x}} \rangle$? Because an inner product is just a sum of pairwise products, it is clear that in order to maximize this sum, you want to multiply numbers of the same sign. Furthermore, to make the largest possible contribution to the sum, you want to pair the largest (magnitude) number in $\vec{x}$ with the largest (magnitude) number in $\tilde{\vec{x}}$. Then, you want to pair the second largest with the second largest, the third with the third, and so on. Continuing in this fashion, you find that the $\tilde{\vec{x}}$ that maximizes the inner product is simply $\vec{x}$. Any other arrangement will produce an inner product of lesser or equal absolute value. Because circular shifts are just a special case of rearrangement, we have proved the claim.

3. **Search and Rescue Dogs**

Berkeley's Puppy Pound needs your help! While Mr. Muffin was being walked, the volunteer let go of his leash and he is now running wild in the streets of Berkeley (which are quite dangerous)! Thankfully, all of the puppies at the pound have a collar that sends a bluetooth signal to receiver towers, which are spread throughout the streets (pictured below). If the puppy/collar is within range of the receiver tower, the collar will send the tower a message: the distance of the collar to the tower. Each cell tower has a range of 3 city blocks. Can you help the pound locate their lost puppy?

**Note**: A city block is defined as the middle of an intersection to the middle of an adjacent intersection (scale provided on map.) Mr. Muffin is constrained to running wild in the streets, meaning he won't be found in any buildings. If your TA asks 'Where is Mr. Muffin?' it is sufficient to answer with his intersection or 'between these two intersections.'

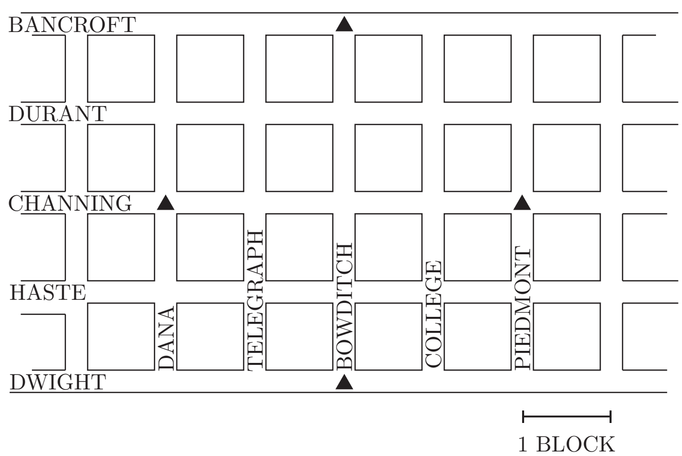
 [1]

(a) You check the logs of the cell towers, and they have received the following messages:

| Sensor | Distance |
|---|---|
| N | 1.3 |
| W | 3 |
| E | 1.5 |
| S | 3 |

On the map provided, identify where Mr. Muffin is!

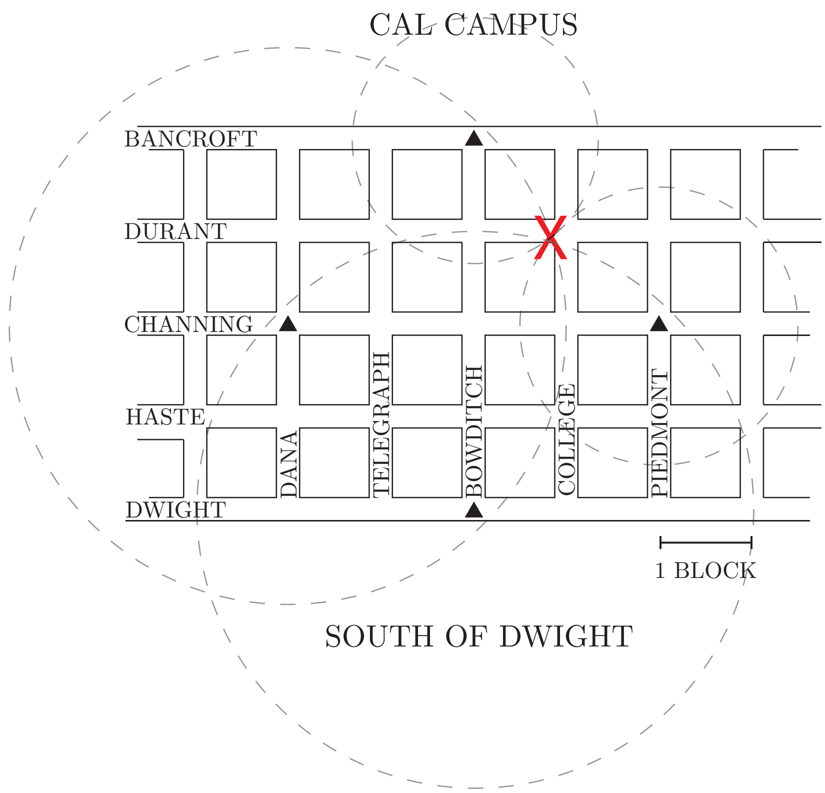
CAL CAMPUS




SOUTH OF DWIGHT

**Answer:**

[1] http://www.pupsmile.com/wp-content/uploads/2012/11/running_happy_dog-1024x684.jpeg




(b) Can you set this up as a system of equations? Is it linear? If it's not linear, can you think of a way to make it linear? Now, how do you set this up in matrix form?

*Hint:* Set $(0,0)$ to be Channing and Bowditch.

*Hint 2:* Distance $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

*Hint 3:* You don't need all 4 equations. You have two unknowns, $x$ and $y$. You know from lecture that you need three circles to uniquely find a point. How can you use the third circle/equation to get two equations and two unknowns?

**Answer:**

First, set up the system of equations:

$$(x-0)^2 + (y-2)^2 = 1.3^2$$
$$(x+2)^2 + (y-0)^2 = 3.0^2$$
$$(x-2)^2 + (y-0)^2 = 1.5^2$$

Simplify out:

$$x^2 + y^2 - 4y + 4 = 1.3^2$$
$$x^2 + 4x + 4 + y^2 = 3.0^2$$
$$x^2 - 4x + 4 + y^2 = 1.5^2$$

Then subtract equation (1) from equations (2) and (3):

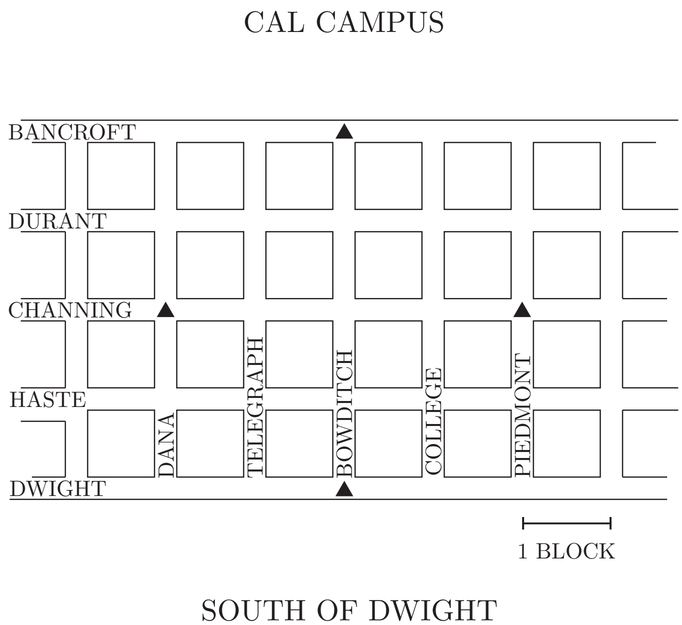
$$4x + 4y = 3.0^2 - 1.3^2$$
$$-4x + 4y = 1.5^2 - 1.3^2$$

This solves to $x = 0.84, y = 0.98$ which is roughly College and Durant.

(c) Suppose Mr. Muffin is moving fast, and by the time you get to destination in part (a) he's already run off! You check the logs of the cell towers again, and see the following updated messages:

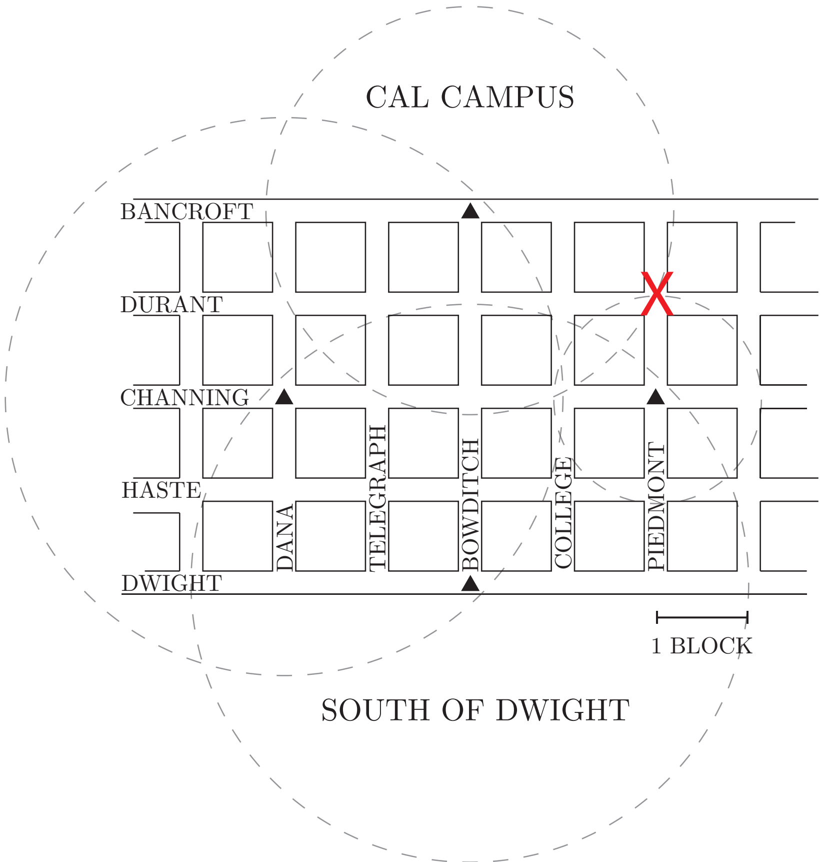
| Sensor | Distance |
|---|---|
| N | 2.2 |
| W | Out of Range |
| E | 1.1 |
| S | Out of Range |

Can you find Mr. Muffin?




**Answer:**

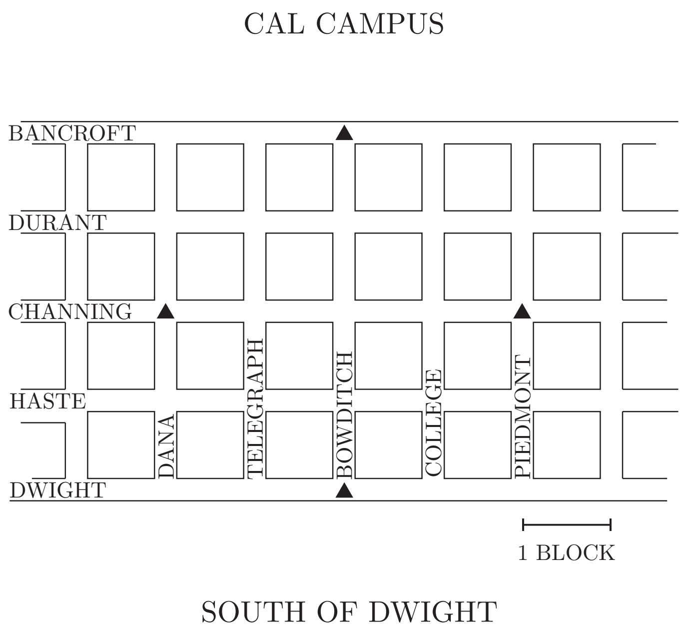
With two out of range sensors, you might think that you will not be able to find a unique solution (you need 3 circles to intersect at a point.) The trick is that out of range still provide information on where Mr. Muffin is NOT. See the diagram below.




(d) Mr. Muffin is a very mischievous puppy, and while playing and running around he damaged his collar. The transmitter on his collar will still send a signal to the receiver towers, but the distance sensor has noise. You check the logs of the cell towers, and they have received the following messages:

| Sensor | Distance |
|---|---|
| N | $1.7 \pm 0.5$ |
| W | $2.1 \pm 0.2$ |
| E | Out of Range |
| S | Out of Range |

On the map provided, identify where Mr. Muffin is! Can you find exactly where he is?




**Answer:**

You can't find exactly where he is, but you know he is somewhere between Dana/Telegraph and Bancroft. See the diagram below.