1. Mechanical Projection

In $\mathbb{R}^n$, the projection of vector $\vec{a}$ onto vector $\vec{b}$ is defined as:

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\|\vec{b}\right\|^2}\vec{b} = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\|\vec{b}\right\|}\hat{b}$$

where $\hat{b}$ is the normalized $\vec{b}$, i.e., a unit vector with the same direction as $\vec{b}$.

(a) Project $\begin{bmatrix}5\\2\end{bmatrix}$ onto $\begin{bmatrix}1\\0\end{bmatrix}$ – that is, onto the x-axis. Graph these two vectors and the projection.

Answer: $\begin{bmatrix}5\\0\end{bmatrix}$

(b) Project $\begin{bmatrix}5\\2\end{bmatrix}$ onto $\begin{bmatrix}0\\1\end{bmatrix}$ – that is, onto the y-axis. Graph these two vectors and the projection.

Answer: $\begin{bmatrix}0\\2\end{bmatrix}$

(c) Project $\begin{bmatrix}4\\-2\end{bmatrix}$ onto $\begin{bmatrix}2\\-1\end{bmatrix}$. Graph these two vectors and the projection.

Answer: $\begin{bmatrix}4\\-2\end{bmatrix}$

(d) Project $\begin{bmatrix}4\\-2\end{bmatrix}$ onto $\begin{bmatrix}1\\1\end{bmatrix}$. Graph these two vectors and the projection.

Answer: $\begin{bmatrix}1\\1\end{bmatrix}$

(e) Project $\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto the span of the vectors $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ – that is, onto the x-y plane in $\mathbb{R}^3$.

Answer: $\begin{bmatrix}1\\2\\0\end{bmatrix}$

(f) Project $\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto the plane described by $x+y+z=1$.

Answer:
Shift the plane to the origin so that we are working with a vector space. $(x-1)+y+z=0=x\prime+y+z$. Using the null space, we can describe this space with the vectors $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$. This is an

orthogonal basis, so the projection is the sum of the independent projections onto each of the vectors. The projection onto the space then is $\begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(g) What is the geometric/physical interpretation of projection? Justify using the previous parts.

Answer:

Geometrically, the orthogonal projection yields the closest vector in the projection space, such that the difference between the original vector and the projected vector is orthogonal to the projected vector. That is, the projection is the best approximation for the original vector within the subspace contraint.

(h) For the first 4 parts, we looked at two different projections for each vector. For those cases, using only the projected vectors and the vectors we projected onto, do we have enough information to reconstruct the original vector?

Answer:

Yes in these cases. The vectors that we projected onto were linearly independent, so we have enough information to reconstruct.

(i) Given information about n projections of a vector in $\mathbb{R}^n$, when do we have enough information to reconstruct the original vector? Always? Never?

Answer:

As stated above, we need those n projections to be on n linearly independent vectors. If these vectors that we are projecting onto are orthogonal, we don't even need to know the vectors; we only need the projections. If they are not orthogonal, we will need the vectors used for projection.

2. Least Squares: A Toy Example

Let's start off by solving a little example of least squares.

We're given the following system of equations:

$$\begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Why can we not solve for $\vec{x}$ exactly?

Answer:

Recall from the earlier linear algebra module that in order for there to be a solution for the matrix system $\mathbf{A}\vec{x} = \vec{b}$, we must have $\vec{b} \in \text{Col}(\mathbf{A})$.

Here, $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. However, we can see that in this case we have $\vec{b} \notin \text{Col}(\mathbf{A})$ because looking at the last row of $\mathbf{A}$, we see that there does not exist a vector $\vec{x}$, such that $\begin{bmatrix} 0 & 0 \end{bmatrix} \vec{x} = 2$.

(b) Find $\vec{\hat{x}}$, the *least squares estimate* of $\vec{x}$, using the formula we derived in lecture.

Answer:

Recall the equation to find the linear least squares estimate:

$$\vec{\hat{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\vec{b}$$

Plugging in $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, we get $\vec{\hat{x}} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$.