1. Orthogonal Matching Pursuit Lecture

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:

- A set of m songs, each of length n: $\mathbf{S} = \{\vec{S}_0, \vec{S}_1, ..., \vec{S}_{m-1}\}$
- An n-dimensional received signal vector: $\vec{r}$
- The sparsity level k of the signal
- Some threshold, th. When the norm of the signal is below this value, the signal contains only noise.

Outputs:

- A set of songs that were identified, F, which will contain at most k elements
- A vector $\vec{x}$ containing song messages $(a_1, a_2, \ldots)$, which will be of length k or less
- An n-dimensional residual $\vec{y}$

Procedure:

- Initialize the following values: $\vec{y} = \vec{r}$, $j = 1$, k, $\mathbf{A} = \begin{bmatrix} \quad \end{bmatrix}$, $F = \{\emptyset\}$
- while (($j \leq k$) and ($\| \vec{y} \| \geq th$)) :
  (a) Cross-correlate $\vec{y}$ with the shifted versions of all songs. Find the song index i and the shifted version of the song, $\vec{S}_i^N$, with which the received signal has the highest correlation value.
  (b) Add i to the set of song indices F.
  (c) Column concatenate matrix $\mathbf{A}$ with the correctly shifted version of the song: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & \mid & \vec{S}_i^N \end{bmatrix}$
  (d) Use least squares to obtain the message value: $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{r}$
  (e) Update the residual value $\vec{y}$ by subtracting: $\vec{y} = \vec{r} - \mathbf{A}\vec{x}$
  (f) Update the counter: $j = j + 1$

2. One Magical Procedure (Fall 2015 Final)

Suppose that we have a vector $\vec{x} \in \mathbb{R}^5$ and an $N \times 5$ measurement matrix $\mathbf{M}$ defined by column vectors $\vec{c}_1, \ldots, \vec{c}_5$, such that:

$$\mathbf{M}\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^N$ as a noisy measurement of the vector $\vec{x}$, with measurement matrix $\mathbf{M}$ and some additional noise in it as well.

You also know that the true $\vec{x}$ is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original $\vec{x}$ as best we can.

However, your intern has managed to lose not only the measurements $\vec{b}$ but the entire measurement matrix $\mathbf{M}$ as well!

Fortunately, you have found a backup in which you have all the pairwise inner products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of $\mathbf{M}$ and each other as well as all the inner products $\left\langle \vec{c}_i, \vec{b} \right\rangle$ between the columns of $\mathbf{M}$ and the vector $\vec{b}$. Finally, you also know the inner product $\left\langle \vec{b}, \vec{b} \right\rangle$ of $\vec{b}$ with itself.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

$\langle \cdot, \cdot \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, that the inner product $\left\langle \vec{c}_3, \vec{b} \right\rangle = 2$, and that the inner product $\left\langle \vec{b}, \vec{b} \right\rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of $\vec{x}$ are non-zero and what their values are.

(a) Use the information in the table above to answer which of the $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with $\vec{b}$.

Answer:

Reading off the table, $\boxed{\vec{c}_4}$ has the largest inner product with $\vec{b}$.

(b) Let the vector with the largest magnitude inner product with $\vec{b}$ be $\vec{c}_a$. Let $\vec{b}_p$ be the projection of $\vec{b}$ onto $\vec{c}_a$. Write $\vec{b}_p$ symbolically as an expression only involving $\vec{c}_a$, $\vec{b}$, and their inner products with themselves and each other.

Answer:

The magnitude of the projection is $\frac{\langle \vec{c}_a, \vec{b} \rangle}{\|\vec{c}_a\|}$, and the direction of the projection is $\frac{\vec{c}_a}{\|\vec{c}_a\|}$. Thus:

$$\vec{b}_p = \boxed{\frac{\left\langle \vec{c}_a, \vec{b} \right\rangle}{\langle \vec{c}_a, \vec{c}_a \rangle} \vec{c}_a}$$

(c) Use the information in the table above to find which of the column vectors $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.

Hint: The linearity of inner products might prove useful.

Answer:

The inner product of $\vec{b} - \vec{b}_p$ with a vector $\vec{c}_i$ is:

$$\left\langle \vec{b} - \vec{b}_p, \vec{c}_i \right\rangle = \left\langle \vec{b}, \vec{c}_i \right\rangle - \frac{\left\langle \vec{c}_a, \vec{b} \right\rangle}{\langle \vec{c}_a, \vec{c}_a \rangle} \langle \vec{c}_a, \vec{c}_i \rangle$$

Finding the numerical values of the inner products:

$$\left\langle \vec{b}-\vec{b}_p, \vec{c}_1 \right\rangle \quad \left\langle \vec{b}-\vec{b}_p, \vec{c}_2 \right\rangle \quad \left\langle \vec{b}-\vec{b}_p, \vec{c}_3 \right\rangle \quad \left\langle \vec{b}-\vec{b}_p, \vec{c}_4 \right\rangle \quad \left\langle \vec{b}-\vec{b}_p, \vec{c}_5 \right\rangle$$

$$4 \qquad -2 \qquad 2 \qquad 0 \qquad 2$$

Thus the vector with the highest inner product with the residue is: $\boxed{\vec{c}_1}$.

(d) Suppose that the vectors we found in parts (a) and (c) are $\vec{c}_a$ and $\vec{c}_c$. These correspond to the components of $\vec{x}$ that are non-zero, that is, $\vec{b} \approx x_a\vec{c}_a + x_c\vec{c}_c$. However, there might be noise in the measurements $\vec{b}$, so we want to find the linear least squares estimates $\widehat{x}_a$ and $\widehat{x}_c$. Write a matrix expression for $\begin{bmatrix} \widehat{x}_a \\ \widehat{x}_c \end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a$, $\vec{c}_c$, $\vec{b}$.

Answer:

We use least squares to solve for $\begin{bmatrix} \widehat{x}_a \\ \widehat{x}_c \end{bmatrix}$. Let $\mathbf{A} = \begin{bmatrix} \vec{c}_a & \vec{c}_c \end{bmatrix}$. Using the least-squares formula,

$$\begin{bmatrix} \widehat{x}_a \\ \widehat{x}_c \end{bmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$$

$$= \begin{bmatrix} \langle \vec{c}_a, \vec{c}_a \rangle & \langle \vec{c}_a, \vec{c}_c \rangle \\ \langle \vec{c}_c, \vec{c}_a \rangle & \langle \vec{c}_c, \vec{c}_c \rangle \end{bmatrix}^{-1} \begin{bmatrix} \left\langle \vec{c}_a, \vec{b} \right\rangle \\ \left\langle \vec{c}_c, \vec{b} \right\rangle \end{bmatrix}$$

(e) Compute the numerical values of $\widehat{x}_a$ and $\widehat{x}_c$ using the information in the table.

Answer:

Substituting the previous expression with values from the table, we get: $x_1 = 2\frac{2}{3}, x_4 = 4\frac{1}{3}$.

$$\begin{bmatrix} \widehat{x}_4 \\ \widehat{x}_1 \end{bmatrix} = \begin{bmatrix} \langle \vec{c}_4, \vec{c}_4 \rangle & \langle \vec{c}_4, \vec{c}_1 \rangle \\ \langle \vec{c}_1, \vec{c}_4 \rangle & \langle \vec{c}_1, \vec{c}_1 \rangle \end{bmatrix}^{-1} \begin{bmatrix} \left\langle \vec{c}_4, \vec{b} \right\rangle \\ \left\langle \vec{c}_1, \vec{b} \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{3} \\ \frac{8}{3} \end{bmatrix}$$