1. Mechanical Gram-Schmidt (Fall 2016 Final)

(a) Use Gram-Schmidt to find an orthonormal basis for the following three vectors.

$$\vec{v}_1 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

Answer:

A valid basis $\mathscr{B}$ is given by:

$$\mathscr{B} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$$

(b) Express $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ as vectors in the basis you found in part (a).

Answer:

Using the basis above:

$$[\vec{v}_1]_{\mathscr{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, [\vec{v}_2]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [\vec{v}_3]_{\mathscr{B}} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

2. Gram-Schmidt Properties

(a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the following set of vectors.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and then in the order $\vec{v}_3$, $\vec{v}_2$, $\vec{v}_1$. Do you get the same answer?

Answer:

If we start with $\vec{v}_1$, we get the basis vectors:

$$\vec{e}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \vec{v}_2 - \frac{\langle \vec{e}_1, \vec{v}_2 \rangle}{\|\vec{e}_1\|^2}\vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_3 = \vec{v}_3 - \frac{\langle \vec{e}_1, \vec{v}_3 \rangle}{\|\vec{e}_1\|^2}\vec{e}_1 - \frac{\langle \vec{e}_2, \vec{v}_3 \rangle}{\|\vec{e}_2\|^2}\vec{e}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

If we write the basis starting with $\vec{v}_3$, we get the basis vectors:

$$\vec{e}_1 = \vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\vec{e}_2 = \vec{v}_2 - \frac{\langle \vec{e}_1, , \vec{v}_2 \rangle}{\|\vec{e}_1\|^2}\vec{e}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}$$

$$\vec{e}_3 = \vec{v}_1 - \frac{\langle \vec{e}_1, \vec{v}_1 \rangle}{\|\vec{e}_1\|^2}\vec{e}_1 - \frac{\langle \vec{e}_2, \vec{v}_1 \rangle}{\|\vec{e}_2\|^2}\vec{e}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix}$$

Normalized, this is:

$$\vec{e}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{e}_2 = \sqrt{\frac{3}{2}}\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}, \vec{e}_3 = \sqrt{2}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix}$$

(b) What happens when we perform Gram-Schmidt on a set of n vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$, where only $n-1$ of them are linearly independent?

Answer:

This can be explained geometrically. Remember that Gram-Schmidt is basically about finding the "error" vector between a vector and its projection onto a subspace. If the vector is already in the subspace, then the "error" vector is $\vec{0}$.

3. Orthonormal Projections

(a) Suppose that the $n \times m$ matrix $\mathbf{A}$ has linearly indpendent columns. The vector $\vec{y}$ in $\mathbb{R}^n$ is not in the subspace spanned by the columns of $\mathbf{A}$. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$ is $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$.

Answer:

When finding a projection onto a subspace, we're trying to find the "closest" vector in that subspace. This can be found by first finding $\vec{x}$ that minimizes $\|\vec{y} - \mathbf{A}\vec{x}\|$. From least squares, we know that $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$. The projection of $\vec{y}$ onto the columns of $\mathbf{A}$ is then $\mathbf{A}\vec{x} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$.

(b) Now suppose that we perform Gram-Schmidt on $\mathbf{A}$ to get a new matrix $\mathbf{Q}$. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{Q}$ is now $\mathbf{Q}\mathbf{Q}^T\vec{y}$.

Answer:

Plugging in $\mathbf{Q}$ for $\mathbf{A}$ in the above expression, we get

$$\mathbf{Q}(\mathbf{Q}^T\mathbf{Q})\mathbf{Q}^T\vec{y} = \mathbf{Q}\mathbf{Q}^T\vec{y}$$

Since we performed Gram-Schmidt on the columns of $\mathbf{A}$ to get $\mathbf{Q}$, we know that $\mathbf{Q}$ is a matrix with orthonormal columns, so $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$.