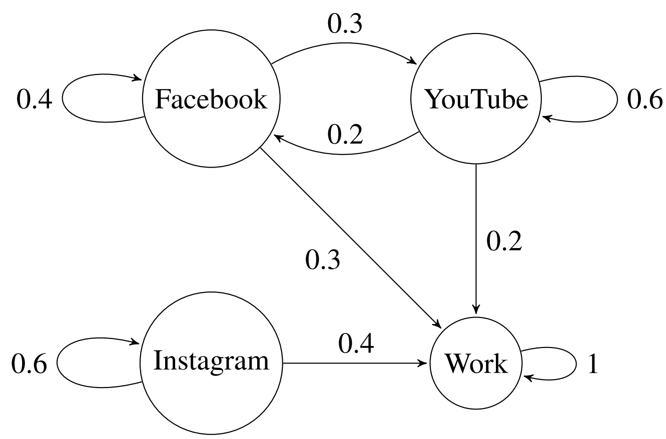
**1. Social Media**

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.




(a) What is the corresponding transition matrix?

**Answer:**

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}$$

(b) There are 150 of you in the class. Suppose on a given Sunday evening (the day when HW is due), there are 70 EE16A students on Facebook, 45 on YouTube, 20 on Instagram, and 15 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

**Answer:**

$$\begin{bmatrix} 37 \\ 48 \\ 12 \\ 53 \end{bmatrix}$$

(c) If the entries in each of the column vectors of your state transition matrix summed to 1, what would this mean with respect to the students on social media? (What is the physical interpretation?)

**Answer:**

We aren't losing students—that is, at a given timestep, a student on a given website either stays on the same website or travels to a different website. No students "disappear," and at the end of many timesteps, we would still have 150 students in the system! This is good—we don't want to be losing students as the semester progresses!

(d) You want to predict how many students will be on each website $n$ timesteps in the future. How would you formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 1000 timesteps/days in the future?

**Answer:**

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}^n \vec{x}[0] = \vec{x}[n]$$

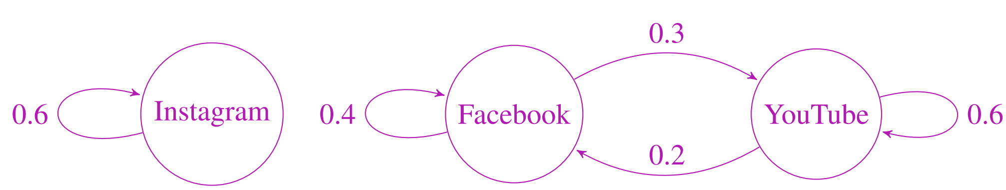
All of them will be working! Yay! With this particular system, 'Work' is called a 'final accepting state' or an 'absorbing state.' This means all the students, after jumping around and being distracted for some amount of time, will eventually end up working. Why is this? 'Work' is the only state where 100% of students who are working remain working. So as time passes, a student has some probability of landing in Work but 0 probability of leaving Work. If you actually calculate $\mathbf{A}^{100}$, you'll see that all the "mass" in the problem transfers to the bottom row, numerically reflecting the fact that 'Work' is absorbing all of the students.

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}^{100} = \begin{bmatrix} 6.83599885 \cdot 10^{-13} & 8.30745059 \cdot 10^{-13} & 0 & 0 \\ 1.24611759 \cdot 10^{-12} & 1.51434494 \cdot 10^{-12} & 0 & 0 \\ 0 & 0 & 6.53318624 \cdot 10^{-23} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The above was calculated using IPython notebook.

(e) **Challenging Practice Problem:** Suppose, instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void," and if a student is "leaked" from any of the other states, we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?

**Answer:**




$$\begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.3 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

Since we don't track students who have gone to work, the entries in the columns of the state transition matrix no longer sum to 1.

**2. Inverses**

In general, the *inverse* of a matrix "undoes" the operation that the matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

**Properties of Inverses**

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\begin{aligned}
&\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \\
&\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} \\
&(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1} \qquad \text{for a nonzero scalar } k \\
&\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T} \\
&(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{assuming } \mathbf{A}, \mathbf{B} \text{ are both invertible}
\end{aligned}$$

(a) Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

**Answer:**

$$\begin{aligned}
\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{ABC} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{IBC} \\
&= \mathbf{C}^{-1}\mathbf{IC} \\
&= \mathbf{I}
\end{aligned}$$

(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$? (*Bonus:* How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)
v. Are all square matrices invertible?
vi. How can you find the inverse of a general $n \times n$ matrix?

**Answer:**

i.
- $\mathbf{A}$: Keeps the $x$ component and throws away the $y$ component.
- $\mathbf{B}$: Keeps the $y$ component and throws away the $x$ component.
- $\mathbf{C}$: Replaces the $x$ and $y$ components with the average of the $x$ and $y$ components.
- $\mathbf{D}$: Finds a weighted sum of the $x$ and $y$ components. Places the sum in $x$ and twice the sum in $y$.

ii. Intuitively, none of these operations can be undone because we lost some information. In the first two, we lost one component of the original. In the third case, we replaced both $x$ and $y$ with the average of the two. Thus, different inputs could lead to the same average and we wouldn't be to tell them apart. In the fourth case, we took a weighted sum of the $x$ and $y$ components. There are different values for $x$ and $y$ that could lead to the same sum. However, we cannot recover the original $x$ and $y$ because we didn't compute two unique weighted sums. Instead, we just multiplied the sum by two for the $y$ component of the output.

iii. Since the operations are not one-to-one reversible, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are not invertible.

iv. The rows of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are all linearly dependent. The same is true for the columns. The generalization is that if a matrix is not invertible, then its rows and columns will be linearly dependent.

v. No. We have seen in the above parts that there are square matrices that are not invertible.

vi. We know that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. If we treat this as our now familiar $\mathbf{A}\vec{x} = \vec{b}$, we can use Gaussian elimination:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right] \implies \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array}\right]$$