1. Mechanical Inverses

In each part, determine whether the inverse of $\mathbf{A}$ exists. If it exists, find it.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow \frac{1}{9}R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow -5R_1 + R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -5 \end{array}\right] \overset{R_1 \leftarrow R_1 + R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & -1 & 1 & -5 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 5 \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$.

(c) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow \frac{1}{5}R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow \frac{1}{2}R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \qquad \overset{R_2 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right] \qquad \overset{R_3 \leftarrow -R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow R_2 + R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right] \qquad \overset{R_1 \leftarrow R_1 - 3R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{5} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{5} & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{4}{5} & 2 & 1 \\ \frac{2}{5} & -\frac{1}{2} & -1 \\ \frac{1}{5} & -\frac{1}{2} & 0 \end{bmatrix}$.

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \qquad \overset{R_1 \leftarrow \frac{1}{5}R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

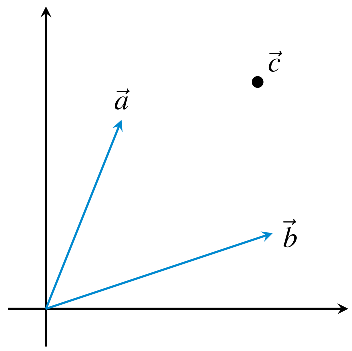
$$\overset{R_2 \leftarrow \frac{1}{2}R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \qquad \overset{R_2 \leftarrow R_2 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \qquad \overset{R_3 \leftarrow R_3 + R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right]$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

2. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a} + \beta\vec{b} = \vec{c}$.



(a) First, consider the case where $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\1\end{bmatrix}$, and $\vec{c} = \begin{bmatrix}-2\\2\end{bmatrix}$. Find the two scalars α and β, such that we reach point $\vec{c}$. What if $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}2\\1\end{bmatrix}$?

Answer:

First set: $\alpha = -4, \beta = 2$
Second set: $\alpha = 6, \beta = -4$

(b) Now formulate the general problem as a system of linear equations and write it in matrix form.

Answer:

$$\begin{cases} \alpha a_1 + \beta b_1 = c_1 \\ \alpha a_2 + \beta b_2 = c_2 \end{cases}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

3. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b)
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$$

In other words, we can swap the order of our spanning vectors and not change their span.

(c)
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change the span.

Answer:

(a) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

Scalar multiplication cancels out. Thus, the spans are the same.

(b) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_2\vec{v}_2 + a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

Swapping the order in addition does not affect the sum, so the spanned spaces are the same.

(c) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, the spanned spaces are the same.