**Reference Definitions**

**Vector spaces:** A *vector space* $V$ is a set of elements that is closed under vector addition and scalar multiplication and contains a zero vector.

That is, if you add two vectors in $V$, your resulting vector will still be in $V$. If you multiply a vector in $V$ by a scalar, your resulting vector will still be in $V$.

**Subspaces:** A subset $W$ of a *vector space* $V$ is a *subspace* of $V$ if the above three conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace $W$.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

**Basis:** A *basis* for a vector space or subspace is an *ordered set of linearly independent vectors* that *spans the vector space or subspace*.

Therefore, if we want to check whether a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ forms a basis for a vector space $V$, we check for two important properties:

(a) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

(b) $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = V$

As we move along, we'll learn how to identify and construct a basis, and we'll also learn some interesting properties of bases.

**Dimension:** The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.

**1. Identifying a Basis**

Does each of these sets of vectors describe a basis for some vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

**Answer:**

- Yes, the vectors are linearly independent, so they are a basis for some 2-dimensional subspace.

- Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$.
- No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent.

**2. Constructing a Basis**

Let's consider a subspace of $\mathbb{R}^3$, $V$, that has the following property: for every vector in $V$, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for $V$. What is the dimension of $V$?

**Answer:**

Any vector $\vec{v}$ in $V$ is going to look as follows:

$$\vec{v} = \begin{bmatrix} 2(a_2 + a_3) \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Now, we consider the set of vectors $\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. The vectors are linearly independent. Furthermore, from the above equation, any vector $\vec{v} \in V$ can be expressed as a linear combination of the vectors in $\mathscr{B}$ (the corresponding coefficients are $a_2$ and $a_3$). This means that $V = \text{span}\{\mathscr{B}\}$.

Therefore,

$$\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for $V$.

$\dim(\mathscr{B}) = 2$ (there are two vectors in $\mathscr{B}$), so the dimension of $V$ is 2.

**3. Exploring Dimension, Linear Independence, and Basis**

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^m$ and a set of $n$ vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

**Answer:**

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^m$. The dimension of $\mathbb{R}^m$ is $m$, so you would need $m$ linearly independent vectors to describe the vector space. Since $n < m$, this is not possible.

(b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

**Answer:**

Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are $m$ of them, they will span $\mathbb{R}^m$.

(c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? What vector space could they form a basis for?

*Hint:* Think about whether the vectors can be linearly independent.

**Answer:**

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^m$. $\mathbb{R}^m$ will be spanned by $m$ linearly independent vectors. Any additional vectors in $\mathbb{R}^m$ must already exist in the span of the previous vectors, and are therefore linearly dependent. Since $n > m$, some of the vectors have to be linearly dependent, so they cannot form a basis.

The two regimes—one where $n > m$ and one where $n < m$—give rise to two different classes of interesting problems. You might learn more about them in upper division courses!

**4. Identifying a Subspace: Proof**

Is the set

$$V = \left\{ \vec{v} \,\middle|\, \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

**Answer:**

Yes, $V$ is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that $V$ is a subset of $\mathbb{R}^3$ – all elements in $V$ are of the form $\begin{bmatrix} c+d \\ c \\ c+d \end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$, such that $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (\alpha d_1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\alpha \vec{v}_1 \in V$.

We have shown both of the no escape (closure) properties, so $V$ is a subspace of $\mathbb{R}^3$.