EECS 16A Designing Information Devices and Systems I
Spring 2018 Discussion 4B

1. Row Operations and Determinants

In this question we explore the effect of row operations on the determinant of a matrix. Note that scaling a row by a will increase the determinant by a factor of a, and adding a multiple of one row to another will not change the determinant. Swapping two rows of a matrix and computing the determinant is equivalent to multiplying the determinant of the original matrix by -1. The determinant of an identity matrix is 1. Feel free to prove these properties to convince yourself that they hold for general square matrices.

(a) An upper triangular matrix is a matrix with zeros below its diagonal. For example a 3×3 upper triangular matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{bmatrix}$$

By considering row operations and what they do to the determinant, argue that the determinant of a general $n \times n$ upper triangular matrix is the product of its diagonal entries if they are non-zero. For example, the determinant of the 3×3 matrix above is $a_1 \cdot b_2 \cdot c_3$ if $a_1, b_2, c_3 \neq 0$.

Answer:

An $n \times n$ upper-triangular matrix looks like:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{bmatrix}$$

For every row i, divide it by $a_{i,i}$. Then we get 1s on the diagonal.

$$\mathbf{A}' = \begin{bmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \cdots & \frac{a_{1,n}}{a_{1,1}} \\ 0 & 1 & \cdots & \frac{a_{2,n}}{a_{2,2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

The determinant of this new matrix has been decreased by a factor of $\frac{1}{a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{n,n}}$:

$$\det(\mathbf{A}') = \frac{\det(\mathbf{A})}{a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{n,n}}$$

Finally, starting from the last row, subtract multiples of the row from the ones above it, so that we get the $n \times n$ identity matrix $\mathbf{I}_n$. This does not change the determinant since we are subtracting rows from each other. Thus:

$$1 = \det(\mathbf{I}_n) = \det(\mathbf{A}') = \frac{\det(\mathbf{A})}{a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{n,n}}$$

$$\det(\mathbf{A}) = a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{n,n}$$

(b) If the diagonal of an upper-triangular matrix has a zero entry, argue that its determinant is still the product of its diagonal entries.

Answer:

If an upper-triangular matrix has a zero in its diagonal, it cannot be row reduced to the identity matrix, which means that that its rows are linearly dependent. Therefore its determinant is zero, which is the product of all diagonal entries (since one of them is zero).

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.

For the eigenvalue $\lambda = 1$:

$$(\mathbf{M} - 1\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_2 = 0$ or equivalently $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$(\mathbf{M} - 9\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = 0$ or equivalently $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

(b) $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\right) = 0$$
$$(1-\lambda)(2-\lambda) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$ and $\lambda = 3$.
For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = -x_1$ or equivalently $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.
For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \vec{x} = \vec{0}$$
$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = 2x_1$ or equivalently $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(c) $\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$

Answer:
Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = 0$$

Without changing the determinant, we can subtract $\frac{3}{\lambda}$ of row 1 from row 2.

$$\det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = 0$$
$$-\lambda(4-\lambda)(3-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$, $\lambda = 3$, and $\lambda = 4$.
For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

which is simply $x_3 = 0$, x_2 is free, and $x_1 = \frac{4}{3}x_2$ or equivalently $\begin{bmatrix} \frac{4}{3}x_2 \\ x_2 \\ 0 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$.

For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -3 & 0 & 0 \\ -3 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = 0$, x_3 is free, and $x_2 = -9x_3$ or equivalently $\begin{bmatrix} 0 \\ -9x_3 \\ x_3 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 0 \\ -9 \\ 1 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 4$:

$$(\mathbf{M} - 4\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -4 & 0 & 0 \\ -3 & 0 & 9 \\ 0 & 0 & -1 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = x_3 = 0$ and $x_2 = x_2$ or equivalently $\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

(d) $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Without changing the determinant, we can add $\frac{1}{\lambda}$ of row 1 to row 2.

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 0 & -\lambda - \frac{1}{\lambda} \end{bmatrix}\right) = 0$$

$$-\lambda\left(-\lambda-\frac{1}{\lambda}\right)=\lambda^2+1=0$$

From the above equation, we know that the eigenvalues are $\lambda = i$ and $\lambda = -i$.
For the eigenvalue $\lambda = i$:

$$\begin{aligned}
(\mathbf{M}-i\mathbf{I})\vec{x} &= \vec{0} \\
\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} - i\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0} \\
\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}\right)\vec{x} &= \vec{0} \\
\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\vec{x} &= \vec{0}
\end{aligned}$$

which is simply $x_1 = ix_2$ and x_2 is free or equivalently $\begin{bmatrix}ix_2\\ x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}i\\ 1\end{bmatrix}\right\}$.
For the eigenvalue $\lambda = -i$:

$$\begin{aligned}
(\mathbf{M}+i\mathbf{I})\vec{x} &= \vec{0} \\
\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + i\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0} \\
\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}\right)\vec{x} &= \vec{0} \\
\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}\vec{x} &= \vec{0}
\end{aligned}$$

which is simply $x_1 = -ix_2$ and x_2 is free or equivalently $\begin{bmatrix}-ix_2\\ x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}-i\\ 1\end{bmatrix}\right\}$.