EECS 16A Designing Information Devices and Systems I
Spring 2018 Discussion 5B

1. Diagonalization

One of the most powerful ways to think about matrices is to think of them in diagonal form [1].

(a) Consider a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

Answer:

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \cdots & \mathbf{A}\vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_k\vec{v}_k \\ | & | & & | \end{bmatrix}$$

$$\mathbf{V\Lambda} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_k\vec{v}_k \\ | & | & & | \end{bmatrix}$$

(b) We now multiply both sides on the right by $\mathbf{V}^{-1}$ and get $\mathbf{A} = \mathbf{V\Lambda V}^{-1}$, the diagonal form of $\mathbf{A}$. Consider the action of $\mathbf{A}$ on a coordinate vector $\vec{x}_u$ in the standard basis. Interpret each step of the following calculation in terms of coordinate transformations and scaling by eigenvalues.

$$\mathbf{A}\vec{x}_u = \mathbf{V\Lambda V}^{-1}\vec{x}_u$$

Answer:

Consider any vector $\vec{x}$ with coordinates $\vec{x}_u$ in the standard basis and $\vec{x}_v$ in a basis composed of the eigenvectors of $\mathbf{A}$. We can think of the action of $\mathbf{A}$ on $\vec{x}_u$ as

$$\mathbf{A}\vec{x}_u = \underbrace{\mathbf{V}\underbrace{\mathbf{\Lambda}\underbrace{\mathbf{V}^{-1}\vec{x}_u}_{\substack{\text{Coords of } \vec{x}_v \\ \text{in eg-vec basis}}}}_{\substack{\text{Coordinates for each eg-vec} \\ \text{scaled by appropriate eg-val}}}}_{\substack{\text{Result transformed} \\ \text{back into standard basis}}}$$

First, the coordinates in the standard basis are transformed into the eigenvector basis ($\mathbf{V}^{-1}\vec{x}_u$). Then the coordinates for each eigenvector are scaled by the appropriate eigenvalue ($\mathbf{\Lambda V}^{-1}\vec{x}_u$). This is the real transformation given by the matrix. Then the result is transformed back into the standard basis. ($\mathbf{V\Lambda V}^{-1}\vec{x}_u$).

[1] Not all matrices can be put in this form but most can. The ones that can't be diagonalized can be put in a similar form called the Jordan form.

2. Matrix Powers

One of the most powerful things about matrix diagonalization is that it gives us some insight into polynomial functions of matrices.

(a) Write $\mathbf{A}^N$ using the diagonalization of $\mathbf{A}$ and simplify your result as much as possible. What do you get?

Answer:

$$\begin{aligned}\mathbf{A}^N &= \underbrace{\mathbf{V}\mathbf{\Lambda}\underbrace{\mathbf{V}^{-1}\times\mathbf{V}}_{\mathbf{I}}\mathbf{\Lambda}\mathbf{V}^{-1}\times\cdots\times\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}}_{N\text{ times}}\\ &= \mathbf{V}\mathbf{\Lambda}^N\mathbf{V}^{-1}\end{aligned}$$

(b) How could you find $\mathbf{A}$ raised to any power while only doing three matrix multiplications?

Answer:

Write $\mathbf{A}^N = \mathbf{V}\mathbf{\Lambda}^N\mathbf{V}^{-1}$. $\mathbf{\Lambda}^N$ can be found by taking each diagonal entry to the Nth power.

(c) Can you suggest an easy way to compute any polynomial function of $\mathbf{A}$?

Answer:

Consider a polynomial function of the following form:

$$\mathbf{P} = \alpha_N\mathbf{A}^N + \alpha_{N-1}\mathbf{A}^{N-1} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

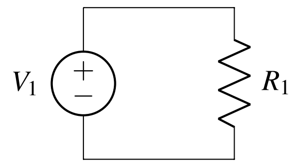
Note that $\mathbf{A}^0 = \mathbf{I}$. We can use the diagonalization of $\mathbf{A}$ as we saw in the previous parts.

$$\begin{aligned}\mathbf{P} &= \alpha_N\mathbf{V}\mathbf{\Lambda}^N\mathbf{V}^{-1} + \alpha_{N-1}\mathbf{V}\mathbf{\Lambda}^{N-1}\mathbf{V}^{-1} + \cdots + \alpha_1\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \alpha_0\mathbf{V}\mathbf{I}\mathbf{V}^{-1}\\ &= \mathbf{V}(\alpha_N\mathbf{\Lambda}^N + \alpha_{N-1}\mathbf{\Lambda}^{N-1} + \cdots + \alpha_1\mathbf{\Lambda} + \alpha_0\mathbf{I})\mathbf{V}^{-1}\end{aligned}$$

In other words, we change $\mathbf{A}$ into the basis of its eigenvectors, compute the polynomial, and then change the result back to the standard basis. Note that since $\mathbf{\Lambda}$ is diagonal, the computation requires much less work.

3. A Simple Circuit

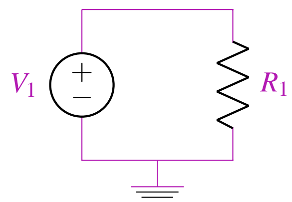
For the circuit shown below, find the voltages across all the elements and the currents through all the elements.



(a) In the above circuit, pick a ground node. Does your choice of ground matter?

Answer:

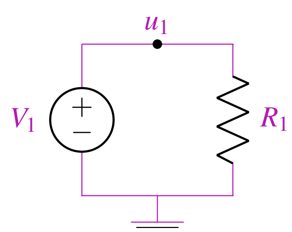
There are two nodes in this circuit and thus two choices for the ground node. The choice of ground does not matter. We will use the ground node shown below:



(b) With your choice of ground, label the node potentials for every node in the circuit.

Answer:

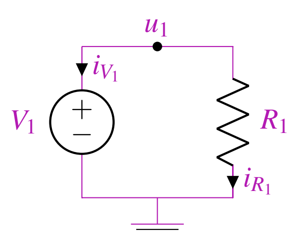
Since this circuit only has two nodes, there will only be one additional node potential.



(c) Label all of the branch currents. Does the direction you pick matter?

Answer:

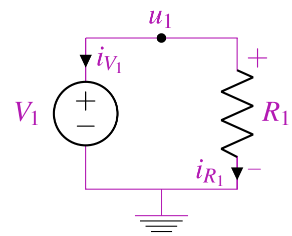
When labeling the currents through branches, the direction you pick does not matter.



(d) Draw the $+/-$ labels on every element. What convention must you follow?

Answer:

When drawing the $+/-$ labels, you must follow the passive sign convention. That is, current flows into the $+$ terminal of every element.



(e) Set up a matrix equation in the form $\mathbf{A}\vec{x} = \vec{b}$ to solve for the unknown node potentials and currents. What are the dimensions of the matrix $\mathbf{A}$?

Answer:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} i_{V_1} \\ i_{R_1} \\ u_1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$\mathbf{A}$ will be a 3×3 matrix since there are three unknowns in the circuit, the two currents i_{V_1} and i_{R_1} and the one potential u_1.

(f) Use KCL to find as many equations as you can for the matrix.

Answer:

KCL gives us one equation for the node at the top, namely that $i_{V_1} + i_{R_1} = 0$. Thus, so far our matrix is as follows:

$$\begin{bmatrix} 1 & 1 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} i_{V_1} \\ i_{R_1} \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ ? \\ ? \end{bmatrix}$$

(g) Use IV relations to find the remaining the equations for the matrix.

Answer:

We can use IV relations to find two more equations. We know that the difference in potentials across the voltage source must be the voltage on the voltage source. We also know that the voltage across the resistor is equal to the current times the resistance. Thus, we have the following equations.

$$u_1 - 0 = V_1$$

$$u_1 - 0 = i_1 R_1 \implies u_1 - i_1 R_1 = 0$$

Our matrix is then:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -R_1 & 1 \end{bmatrix} \begin{bmatrix} i_{V_1} \\ i_{R_1} \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 \\ 0 \end{bmatrix}$$

(h) Solve the system of equations if $V_1 = 5\,\text{V}$ and $R_1 = 5\,\Omega$.

Answer:

By plugging the given values into the system of equations, we get:

$$\begin{bmatrix} i_{V_1} \\ i_{R_1} \\ u_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$