Reference Definitions

Inner products: An inner product is a function that associates each pair of two vectors in a vector space V with a real number (called the inner product). For any $\vec{x}, \vec{y}, \vec{z} \in V$ and $c \in \mathbb{R}$, the inner product satisfies the following three properties:

(a) **Symmetry:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(b) **Linearity:**

i. $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

ii. $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$

(c) **Positive-definiteness:** $\langle \vec{x}, \vec{x} \rangle \geq 0$ with $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$

Norm: The norm of a vector $\vec{x} \in V$ is defined to be:

$$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle \implies \|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

1. From Inner Products To Projections

Given that $\langle \vec{x}, \vec{y} \rangle$ is a measure of similarity between two vectors, let's try to use this to find how much of one vector $\vec{y}$ is in the direction of another vector $\vec{x}$.

(a) Let's start with $\langle \vec{x}, \vec{y} \rangle$. We want a quantity that is independent of the norm of $\vec{x}$, $\|\vec{x}\|$. Is $\langle \vec{x}, \vec{y} \rangle$ independent of the norm? Consider $\langle \vec{x}, \vec{y} \rangle$ for the examples below.
$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) Suppose we divide $\langle \vec{x}, \vec{y} \rangle$ by the norm of $\vec{x}$, $\|\vec{x}\|$, to get $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|}$. Is this new quantity independent of the norm of $\vec{x}$? Test it on the examples above.

(c) We now have a scalar quantity that represents how much of $\vec{y}$ is in the direction of $\vec{x}$. Let's try to find a vector that is how much of $\vec{y}$ is in the $\vec{x}$ direction. That is, we are looking for a vector $\vec{z}$ that has a norm of $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|}$ and points in the same direction as $\vec{x}$.

(d) Given the projection between two vectors, defined as $\text{proj}_{\vec{x}}\vec{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|^2}\vec{x}$, prove the Cauchy-Schwarz inequality, $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$.

(e) Consider the quantity $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|}$. What is the maximum this quantity could be? When does this occur? What is the minimum this quantity could be? When does this occur?

(f) We define the angle between two vectors as $\cos(\theta) = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|}$. When do two vectors have an angle of $90°$ between them? When do they have an angle of $0°$? When do they have an angle of $180°$?

2. Packings

(a) Can three vectors in the $\mathbb{R}^2$ plane have only negative pairwise inner-products? That is, do there exist vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ such that $\langle \vec{u}, \vec{v} \rangle < 0$, $\langle \vec{v}, \vec{w} \rangle < 0$, and $\langle \vec{u}, \vec{w} \rangle < 0$?

Hint: Draw a picture!

(b) What about four vectors in $\mathbb{R}^2$? That is, do there exist four vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^2$ such that for every pair of vectors $\vec{a}, \vec{b}$: $\langle \vec{a}, \vec{b} \rangle < 0$?

Bonus: What about four vectors in $\mathbb{R}^3$?

3. Orthogonal Subspaces

Two vectors are $\vec{x}$ and $\vec{y}$ are said to be orthogonal if their inner product is zero. That is $\langle \vec{x}, \vec{y} \rangle = 0$.

Two subspaces $\mathbb{S}_1$ and $\mathbb{S}_2$ of $\mathbb{R}^N$ are said to be orthogonal if all vectors in $\mathbb{S}_1$ are orthogonal to all vectors in $\mathbb{S}_2$. That is,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = 0 \; \forall \vec{v}_1 \in \mathbb{S}_1, \vec{v}_2 \in \mathbb{S}_2.$$

(a) Recall that the *row space* of an $M \times N$ matrix $\mathbf{A}$ is the subspace spanned by the rows of $\mathbf{A}$ and that the *null space* of $\mathbf{A}$ is the subspace of all vectors $\vec{v}$ such that $\mathbf{A}\vec{v} = \vec{0}$.
Prove that the row space and null space of any matrix are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}^T) \perp \text{Null}(\mathbf{A}) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.

(b) Recall that the *column space* of an $M \times N$ matrix $\mathbf{A}$ is the subspace spanned by the columns of $\mathbf{A}$ and that the *left null space* of $\mathbf{A}$ is the subspace of all vectors $\vec{v}$ such that $\vec{v}^T \mathbf{A} = \vec{0}^T \iff \mathbf{A}^T \vec{v} = \vec{0}$.
Prove that the column space and left null space of any matrix are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}) \perp \text{Null}(\mathbf{A}^T) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.