

**1. Mechanical Projection**

In  $\mathbb{R}^n$ , the projection of vector  $\vec{a}$  onto vector  $\vec{b}$  is defined as:

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{b}\|^2} \vec{b} = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{b}\|} \hat{b}$$

where  $\hat{b}$  is the normalized  $\vec{b}$ , i.e., a unit vector with the same direction as  $\vec{b}$ .

(a) Project  $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$  onto  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  – that is, onto the  $x$ -axis. Graph these two vectors and the projection.

(b) Project  $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$  onto  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  – that is, onto the  $y$ -axis. Graph these two vectors and the projection.

(c) Project  $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$  onto  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ . Graph these two vectors and the projection.

(d) Project  $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$  onto  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Graph these two vectors and the projection.

(e) Project  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  onto the span of the vectors  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  – that is, onto the  $x$ - $y$  plane in  $\mathbb{R}^3$ .

(f) Project  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  onto the plane described by  $x + y + z = 1$ .

(g) What is the geometric/physical interpretation of projection? Justify using the previous parts.

(h) For the first 4 parts, we looked at two different projections for each vector. For those cases, using only the projected vectors and the vectors we projected onto, do we have enough information to reconstruct the original vector?

(i) Given information about  $n$  projections of a vector in  $\mathbb{R}^n$ , when do we have enough information to reconstruct the original vector? Always? Never?

## 2. Least Squares: A Toy Example

Let's start off by solving a little example of least squares.

We're given the following system of equations:

$$\begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

where  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .

- (a) Why can we not solve for  $\vec{x}$  exactly?
- (b) Find  $\vec{\hat{x}}$ , the *least squares estimate* of  $\vec{x}$ , using the formula we derived in lecture.