

1. Orthogonal Matching Pursuit Lecture

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:

- A set of m songs, each of length n : $\mathbf{S} = \{\vec{s}_0, \vec{s}_1, \dots, \vec{s}_{m-1}\}$
- An n -dimensional received signal vector: \vec{r}
- The sparsity level k of the signal
- Some threshold, th . When the norm of the signal is below this value, the signal contains only noise.

Outputs:

- A set of songs that were identified, F , which will contain at most k elements
- A vector \vec{x} containing song messages (a_1, a_2, \dots) , which will be of length k or less
- An n -dimensional residual \vec{y}

Procedure:

- Initialize the following values: $\vec{y} = \vec{r}$, $j = 1$, k , $\mathbf{A} = []$, $F = \{\emptyset\}$
- while $((j \leq k) \text{ and } (\|\vec{y}\| \geq th))$:
 - (a) Cross-correlate \vec{y} with the shifted versions of all songs. Find the song index i and the shifted version of the song, \vec{s}_i^N , with which the received signal has the highest correlation value.
 - (b) Add i to the set of song indices F .
 - (c) Column concatenate matrix \mathbf{A} with the correctly shifted version of the song: $\mathbf{A} = [\mathbf{A} \mid \vec{s}_i^N]$
 - (d) Use least squares to obtain the message value: $\vec{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{r}$
 - (e) Update the residual value \vec{y} by subtracting: $\vec{y} = \vec{r} - \mathbf{A} \vec{x}$
 - (f) Update the counter: $j = j + 1$

2. One Magical Procedure (Fall 2015 Final)

Suppose that we have a vector $\vec{x} \in \mathbb{R}^5$ and an $N \times 5$ measurement matrix \mathbf{M} defined by column vectors $\vec{c}_1, \dots, \vec{c}_5$, such that:

$$\mathbf{M}\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \dots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^N$ as a noisy measurement of the vector \vec{x} , with measurement matrix \mathbf{M} and some additional noise in it as well.

You also know that the true \vec{x} is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original \vec{x} as best we can.

However, your intern has managed to lose not only the measurements \vec{b} but the entire measurement matrix \mathbf{M} as well!

Fortunately, you have found a backup in which you have all the pairwise inner products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of \mathbf{M} and each other as well as all the inner products $\langle \vec{c}_i, \vec{b} \rangle$ between the columns of \mathbf{M} and the vector \vec{b} . Finally, you also know the inner product $\langle \vec{b}, \vec{b} \rangle$ of \vec{b} with itself.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

$\langle \cdot, \cdot \rangle$	\vec{c}_1	\vec{c}_2	\vec{c}_3	\vec{c}_4	\vec{c}_5	\vec{b}
\vec{c}_1	2	0	1	-1	1	1
\vec{c}_2		2	1	-1	-1	-5
\vec{c}_3			2	0	-1	2
\vec{c}_4				2	-1	6
\vec{c}_5					2	-1
\vec{b}						29

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, that the inner product $\langle \vec{c}_3, \vec{b} \rangle = 2$, and that the inner product $\langle \vec{b}, \vec{b} \rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of \vec{x} are non-zero and what their values are.

- (a) Use the information in the table above to answer which of the $\vec{c}_1, \dots, \vec{c}_5$ has the largest magnitude inner product with \vec{b} .
- (b) Let the vector with the largest magnitude inner product with \vec{b} be \vec{c}_a . Let \vec{b}_p be the projection of \vec{b} onto \vec{c}_a . Write \vec{b}_p symbolically as an expression only involving \vec{c}_a, \vec{b} , and their inner products with themselves and each other.
- (c) Use the information in the table above to find which of the column vectors $\vec{c}_1, \dots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.

Hint: The linearity of inner products might prove useful.

- (d) Suppose that the vectors we found in parts (a) and (c) are \vec{c}_a and \vec{c}_c . These correspond to the components of \vec{x} that are non-zero, that is, $\vec{b} \approx x_a \vec{c}_a + x_c \vec{c}_c$. However, there might be noise in the measurements \vec{b} , so we want to find the linear least squares estimates \hat{x}_a and \hat{x}_c . Write a matrix expression for $\begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a, \vec{c}_c, \vec{b}$.
- (e) Compute the numerical values of \hat{x}_a and \hat{x}_c using the information in the table.