

## Reference Definitions

**Vector spaces:** A *vector space*  $V$  is a set of elements that is closed under vector addition and scalar multiplication and contains a zero vector.

That is, if you add two vectors in  $V$ , your resulting vector will still be in  $V$ . If you multiply a vector in  $V$  by a scalar, your resulting vector will still be in  $V$ .

**Subspaces:** A subset  $W$  of a *vector space*  $V$  is a *subspace* of  $V$  if the above three conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace  $W$ .

The vector spaces we will work with most commonly are  $\mathbb{R}^n$  and  $\mathbb{C}^n$  as well as their subspaces.

**Basis:** A *basis* for a vector space or subspace is an *ordered set of linearly independent vectors* that *spans the vector space or subspace*.

Therefore, if we want to check whether a set of vectors  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  forms a basis for a vector space  $V$ , we check for two important properties:

- (a)  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is linearly independent.
- (b)  $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = V$

As we move along, we'll learn how to identify and construct a basis, and we'll also learn some interesting properties of bases.

**Dimension:** The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.

### 1. Identifying a Basis

Does each of these sets of vectors describe a basis for some vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

## 2. Constructing a Basis

Let's consider a subspace of  $\mathbb{R}^3$ ,  $V$ , that has the following property: for every vector in  $V$ , the first entry is equal to two times the sum of the second and third entries. That is, if  $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$ ,  $a_1 = 2(a_2 + a_3)$ .

Find a basis for  $V$ . What is the dimension of  $V$ ?

## 3. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space  $\mathbb{R}^m$  and a set of  $n$  vectors  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  in  $\mathbb{R}^m$ .

- For the first part of the problem, let  $m > n$ . Can  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  form a basis for  $\mathbb{R}^m$ ? Why/why not? What conditions would we need?
- Let  $m = n$ . Can  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  form a basis for  $\mathbb{R}^m$ ? Why/why not? What conditions would we need?
- Now, let  $m < n$ . Can  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  form a basis for  $\mathbb{R}^m$ ? What vector space could they form a basis for?

*Hint:* Think about whether the vectors can be linearly independent.

## 4. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \mid \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of  $\mathbb{R}^3$ ? Why/why not?