EECS 16A Designing Information Devices and Systems I
Spring 2018 Homework 2

This homework is due February 5, 2018, at 23:59.
Self-grades are due February 8, 2018, at 23:59.

Submission Format
Your homework submission should consist of **two** files.

- `hw2.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

- `hw2.ipynb`: A single IPython notebook with all of your code in it.

Submit each file to its respective assignment on Gradescope.

1. (PRACTICE) Powers Of Nilpotent Matrices

Do this problem if you would like more mechanical practice with matrix multiplication.
The following matrices are examples of a special type of matrix called a nilpotent matrix. What happens to each of these matrices when you multiply it by itself four times? Multiply them to find out. Why do you think these are called "nilpotent" matrices? (Of course, there is nothing magical about 4×4 matrices. You can have nilpotent square matrices of any dimension greater than 1.)

(a) Calculate $\mathbf{A}^4$ by hand. Make sure you show what $\mathbf{A}^2$ and $\mathbf{A}^3$ are along the way.

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^4 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Calculate $\mathbf{B}^4$ by hand. Make sure you show what $\mathbf{B}^2$ and $\mathbf{B}^3$ are along the way.

$$\mathbf{B} = \begin{bmatrix} 3 & 4 & 2 & 1 \\ -5 & -6 & -3 & -1 \\ 6 & 7 & 3 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

Solution:

$$\mathbf{B}^2 = \begin{bmatrix} 3 & 4 & 2 & 1 \\ -5 & -6 & -3 & -1 \\ 6 & 7 & 3 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 & 1 \\ -5 & -6 & -3 & -1 \\ 6 & 7 & 3 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 & 3 \\ -5 & -7 & -2 & -5 \\ 5 & 7 & 2 & 5 \\ 2 & 3 & 1 & 2 \end{bmatrix}$$

$$\mathbf{B}^3 = \begin{bmatrix} 3 & 4 & 1 & 3 \\ -5 & -7 & -2 & -5 \\ 5 & 7 & 2 & 5 \\ 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 & 1 \\ -5 & -6 & -3 & -1 \\ 6 & 7 & 3 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 0 & -2 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}^4 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 0 & -2 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 & 1 \\ -5 & -6 & -3 & -1 \\ 6 & 7 & 3 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A nilpotent matrix is a matrix that becomes all 0's when you raise it to some power, i.e. repeatedly multiply it by itself.

2. Elementary Matrices

This week, we learned about an important technique for solving systems of linear equations called Gaussian elimination. It turns out that each row operation in Gaussian elimination can be performed by multiplying the augmented matrix on the left by a specific matrix called an *elementary matrix*. For example, suppose we want to row reduce the following augmented matrix:

$$\mathbf{A} = \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 15 \\ 0 & 1 & 0 & 3 & -7 \\ -2 & -3 & 1 & -6 & 9 \\ 0 & 1 & 0 & 2 & -5 \end{array}\right] \tag{1}$$

What matrix do you get when you subtract the 4th row from the 2nd row of **A** (putting the result in row 2)? (You don't have to include this in your solution.) Now, try multiplying the original **A** on the left by

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(You don't have to include this in your solutions either.) Notice that you get the same thing.

$$\mathbf{EA} = \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 15 \\ 0 & 0 & 0 & 1 & -2 \\ -2 & -3 & 1 & -6 & 9 \\ 0 & 1 & 0 & 2 & -5 \end{array}\right]$$

E is a special type of matrix called an *elementary matrix*. This means that we can obtain the matrix **E** from the identity matrix by applying an elementary row operation – in this case, subtracting the 4th row from the 2nd row.

In general, any elementary row operation can be performed by left multiplying by an appropriate elementary matrix.

In other words, you can perform a row operation on a matrix **A** by first performing that row operation on the identity matrix to get an elementary matrix (see below), and then left multiplying **A** by the elementary matrix (like we did above).

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_4 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$$

(a) Write down the elementary matrices required to perform the following row operations on a 4×5 augmented matrix.

- Switching rows 1 and 3
- Multiplying row 3 by -5
- Adding $3\times$row 2 to row 4 (putting the result in row 4) and subtracting row 2 from row 1 (putting the result in row 1)

*Hint: For the last one, note that if you want to perform two row operations on the matrix **A**, you can perform them both on the identity matrix and then left multiply **A** by the resulting matrix.*

Solution:

We obtain each of the desired elementary matrices by performing the row operations on a 4×4 identity matrix.

- Switching rows 1 and 3:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Multiplying row 3 by -5:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Adding 3×row 2 to row 4 and subtracting row 2 from row 1:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Note that we obtained this last matrix by applying two elementary row operations to the identity matrix. We could have performed each elementary row operation on individual identity matrices and then multiplied them together to achieve the same result. In this case, the order of the matrices did not matter; however, this is not true in general.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

(b) Now, compute a matrix **E** (by hand) that fully row reduces the augmented matrix **A** given in Equation 1 – that is, find **E** such that **EA** is in reduced row echelon form. Show that this is true by multiplying out **EA**. When an augmented matrix is in reduced row echelon form, it will have the form

$$\mathbf{EA} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & b_4 \end{array}\right]$$

Once you have found the required elementary matrices, plug them into the iPython notebook to get the matrix **E**. Verify by hand that multiplying **E** and **A** gives you the identity matrix augmented with constants (as **EA** shown above).

Hint: As before, note that you can either ***apply a set of row operations to the same identity matrix*** *or* ***apply them to separate identity matrices and then multiply the matrices together****. Make sure, though, that you apply the row operations and multiply the matrices in the correct order.*

Solution:

We first need to row reduce **A** by hand to find the required row operations. The following row operations will do the trick (though you could have used a different set that does the same thing.)

- Step 1: Add 2×**Row 1** to Row 3
- Step 2: Add 2×**Row 2** to Row 1, add 7×**Row 2** to Row 3, and subtract **Row 2** from Row 4

- Step 3: Add **Row 4** to Row 1, add $3\times$**Row 4** to Row 2, and add $5\times$**Row 4** to Row 3
- Step 4: Multiply **Row 4** by -1

Note that we have grouped the row operations together, so that each step involves adding a scalar multiple of a particular row to the other rows. This will make calculating the elementary matrices for each step easier.

Applying each of these sets of row operations to a 4×4 identity matrix gives us the following matrices:

- Step 1:

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 2:

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

- Step 3:

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 4:

$$\mathbf{E}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We now multiply these matrices together as follows:

$$\mathbf{E} = \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 3 \\ 2 & 2 & 1 & 5 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Note the order in which we multiplied the matrices. $\mathbf{E}_1$ gets applied first, so it is furthest to the right (i.e. it will act on the augmented matrix first), etc. Also, note that we could have applied the row operations to the identity matrices in different groups. For example, we could have written an elementary matrix for each individual row operation and multiplied all of them together, making sure to maintain the correct order. We also could have applied all of the row operations, in the correct order, to a single identity matrix. **The important thing is that we maintain the correct order of row operations – either when we're applying them to an individual identity matrix or multiplying the elementary matrices together.** You should have done these multiplications in the iPython notebook.

To show that **E** does, in fact, row reduce **A**, we calculate **EA**.

$$\begin{aligned}\mathbf{EA} &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 3 \\ 2 & 2 & 1 & 5 \\ 0 & 1 & 0 & -1 \end{bmatrix} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 15 \\ 0 & 1 & 0 & 3 & -7 \\ -2 & -3 & 1 & -6 & 9 \\ 0 & 1 & 0 & 2 & -5 \end{array}\right] \\ &= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\end{aligned}$$

3. Small angle optics

Physical rules governing optics are non-linear in general. For example, refraction of light is governed by the following law, called Snell's law,

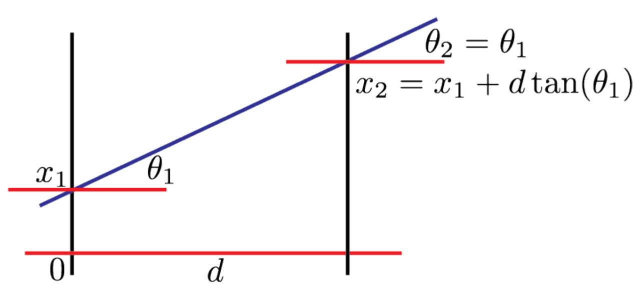
$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2),$$

where n_1 and n_2 are the refractive indices of the input and output media respectively, and θ_1 and θ_2 are the angle the light makes with respect to the normal of the surface, for incoming and outgoing light respectively. As can be seen, the input and output angles are a non-linear function of each other.

However, for small enough angles, we have the following relations, which we call the small angle approximation

$$\begin{aligned}\sin(\theta) &\approx \theta, \\ \cos(\theta) &\approx 1, \\ \tan(\theta) &\approx \theta.\end{aligned}$$

To see how the small angle approximation is used, say we have a light ray at position x_1 and traveling at angle θ_1 with respect to a chosen axis, which we call the optical axis. After the light travels length d with respect to the optical axis, the position and angle become:



$$\begin{aligned}x_2 &= x_1 + d\tan(\theta_1), \\ \theta_2 &= \theta_1.\end{aligned}$$

Using the small angle approximation we have

$$x_2 = x_1 + d\theta_1,$$
$$\theta_2 = \theta_1.$$

Now we can represent these equations as a matrix operation:

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}.$$

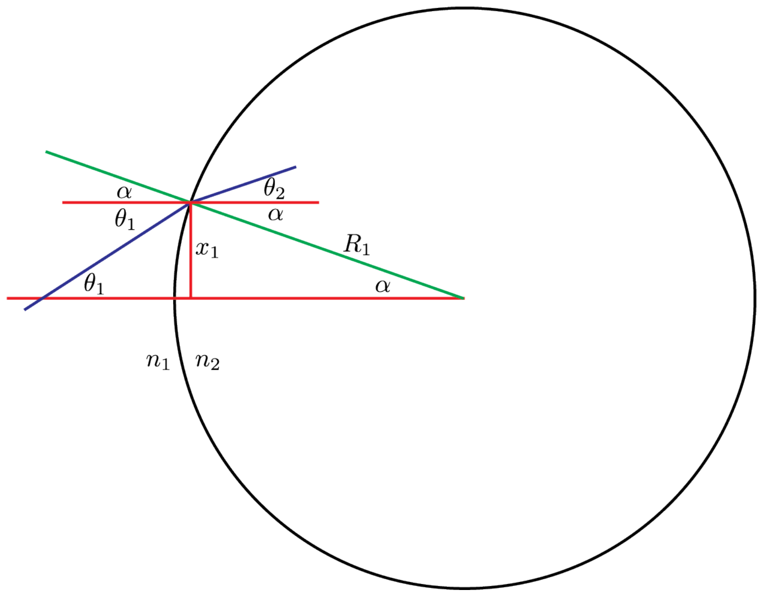


Figure 1: Front surface of the thin lens.

Another common optical event is refraction of light at a spherical surface. (Lenses are frequently have a spherical profile since this is easier to manufacture.) For the refraction at the front surface of a spherical lens, shown in Figure 1 we have

$$x_2 = x_1$$
$$n_1 \sin(\theta_1 + \alpha) = n_2 \sin(\theta_2 + \alpha).$$

Using the small angle approximations, the equation becomes

$$n_1(\theta_1 + \alpha) = n_2(\theta_2 + \alpha)$$

Furthermore we can approximate $\alpha = \sin(\alpha) = x/R_1$

$$\theta_2 = \frac{n_1 - n_2}{n_2 R_1} x_1 + \frac{n_1}{n_2} \theta_1.$$

We can represent this equation, along with the $x_1 = x_2$ equation, in matrix form:

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R_1} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = A_1 \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}.$$

(a) Using similar steps of the above derivation derive the equation for the back curved surface of a spherical lens, shown in Figure 2. That is, find the matrix A_2 giving

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = A_2 \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}.$$

In particular looking at Figure 2, we see that

$$n_2 \sin(\theta_i) = n_1 \sin(\theta_o). \tag{2}$$

Write θ_i as a function of θ_1 and α, and θ_o as a function of θ_2 and α. Substitute inside equation 2 and use the small angle approximations. Then use $\alpha = \sin(\alpha) = x_1/R_2$.

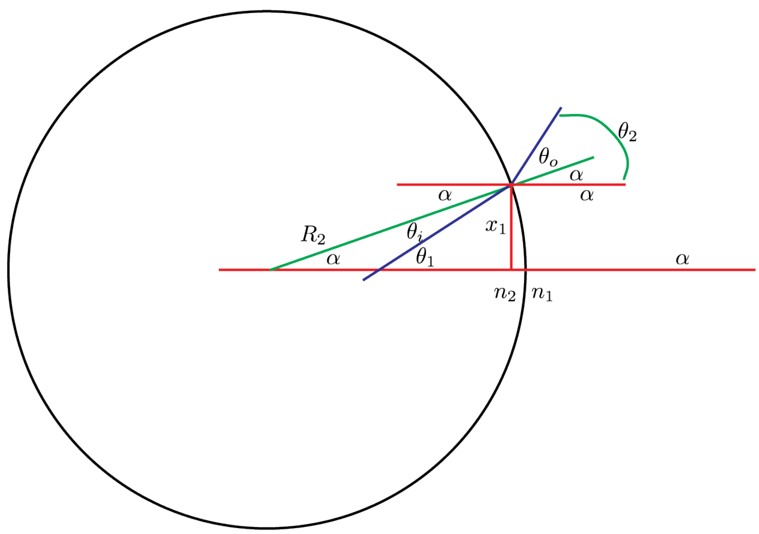


Figure 2: Back surface of the thin lens.

Solution:

$$\theta_i = \theta_1 - \alpha,$$
$$\theta_o = \theta_2 - \alpha,$$

hence we have

$$n_2 \sin(\theta_1 - \alpha) = n_1 \sin(\theta_2 - \alpha).$$

Using small angle approximation we have

$$n_2(\theta_1 - \alpha) = n_1(\theta_2 - \alpha).$$

Using this and substituting for α we have the following system

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_1 R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}.$$

(b) Multiplying the two matrices as A_2A_1, find the transfer function of the thin lens as a whole.

Solution:

$$A_2A_1 = \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_1R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_2R_1} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & 1 \end{bmatrix}.$$

(c) We want to find the focal length of the thin lens. Focal length is the point behind the lens where any ray incident at zero degrees focuses at $x_2 = 0$ independent of x_1. To do that, find d such that x_2 given by the following equation

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} A_2A_1 \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

is exactly 0 irrespective of x_1.

Compare your results to wikipedia article

https://en.wikipedia.org/wiki/Thin_lens

Are you happy with your results?

Solution:

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \frac{n_1-n_2}{n_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)x_1 \end{bmatrix} = \begin{bmatrix} x_1 + d\frac{n_1-n_2}{n_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)x_1 \\ \frac{n_1-n_2}{n_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)x_1. \end{bmatrix}$$

Top line is equal to 0 if

$$d = \frac{n_1}{n_2-n_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-1}.$$

The answer matches the wikipedia article's after the sign convention is applied to R_2, that is wikipedia assumes the second radius is given as a negative value.

4. Show It

Let n be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a set of k linearly dependent vectors in $\mathbb{R}^n$. Show that for any $n \times n$ matrix $\mathbf{A}$, the set $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, \dots, \mathbf{A}\vec{v}_k\}$ is a set of linearly dependent vectors. Make sure that you prove this rigorously for all possible matrices $\mathbf{A}$.

Solution:

It is given that $\{\vec{v}_i \in \mathbb{R}^n | i = 0, 1, \dots, k\}$ is linearly dependent. This implies that there exist k scalars, $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ from $\mathbb{R}$ that are *not all equal to zero simultaneously* (or equivalently *at least one of which is not equal to zero*), such that

$$\alpha_1 \cdot \vec{v}_1 + \alpha_2 \cdot \vec{v}_2 + \cdots + \alpha_k \cdot \vec{v}_k = \vec{0}. \quad (3)$$

By left-multiplying Equation 3 with $\mathbf{A}$, we get

$$\mathbf{A}(\alpha_1 \cdot \vec{v}_1 + \alpha_2 \cdot \vec{v}_2 + \cdots + \alpha_k \cdot \vec{v}_k) = \mathbf{A}\vec{0}.$$

First, note that $\mathbf{A}\vec{0} = \vec{0}$. Moreover, if we distribute, we get

$$\mathbf{A}(\alpha_1 \cdot \vec{v}_1) + \mathbf{A}(\alpha_2 \cdot \vec{v}_2) + \cdots + \mathbf{A}(\alpha_k \cdot \vec{v}_k) = \vec{0}.$$

From associativity of multiplication, we get

$$(\mathbf{A}\alpha_1)\vec{v}_1 + (\mathbf{A}\alpha_2)\vec{v}_2 + \cdots + (\mathbf{A}\alpha_k)\vec{v}_k = \vec{0}.$$

Since scalar-matrix multiplication is commutative, we get

$$(\alpha_1\mathbf{A})\vec{v}_1 + (\alpha_2\mathbf{A})\vec{v}_2 + \cdots + (\alpha_k\mathbf{A})\vec{v}_k = \vec{0}.$$

By using associativity of multiplication again, we get

$$\alpha_1 \cdot (\mathbf{A}\vec{v}_1) + \alpha_2 \cdot (\mathbf{A}\vec{v}_2) + \cdots + \alpha_k \cdot (\mathbf{A}\vec{v}_k) = \vec{0}. \tag{4}$$

Therefore, the same k scalars, $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, show the linear dependence of the vectors $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, \ldots, \mathbf{A}\vec{v}_k\}$, as requested.

■

Note: There are alternative and equivalent implications of linear dependence that can be used in the proof (instead of Equation 3). Here are a few of them:

(a) Some vector $\vec{v}_j$ can be represented as a linear combination of the *other* vectors as follows: There exist scalars $\alpha_i, 1 \leq i \leq k, i \neq j$, such that

$$\sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i \cdot \vec{v}_i = \vec{v}_j.$$

(In this alternative, the scalars may all be zeros, and the linear combination on the left hand side must exclude $\vec{v}_j$.)

(b) There exists $1 \leq j \leq k$ such that $\vec{v}_j$ can be represented as a linear combination of the $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$ as follows: There exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}$, such that

$$\sum_{i=1}^{j-1} \alpha_i \cdot \vec{v}_i = \vec{v}_j.$$

(In this alternative, the scalars may all be zeros, and the linear combination on the left hand side must exclude $\vec{v}_j$.)

For all of these alternatives, the rest of the proof is similar to the one demonstrated above (multiply both sides by $\mathbf{A}$ and then apply the linearity of matrix multiplication to include the $\mathbf{A}$ in the relevant sum and next to the $\vec{v}_i$) and will result in an equation similar to Equation 4 (that matches the chosen alternative).

Common mistakes included:

- When using the definition provided in Equation 3, not indicating that the at least one of scalars needs to be non-zero.

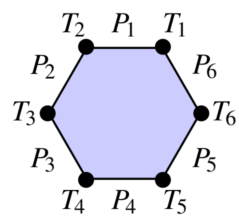
- When using the definition provided in Equation 3, stating that all scalars need to be non-zero.
- When using any of the alternative implications (a)-(b) above, requiring that at least one scalar be non-zero (or all non-zero).
- When using any of the alternative implications (a)-(b) above, not excluding the vector on the right hand side from the linear combination on the left hand side.

5. Figuring Out The Tips

A number of people gather around a round table for a dinner. Between every adjacent pair of people, there is a plate for tips. When everyone has finished eating, each person places half their tip in the plate to their left and half in the plate to their right. In the end, of the tips in each plate, some of it is contributed by the person to its right, and the rest is contributed by the person to its left. Suppose you can only see the plates of tips after everyone has left. Can you deduce everyone's individual tip amounts?

Note: For this question, if we assume that tips are positive, we need to introduce additional constraints enforcing that, and we wouldn't get a linear system of equations. Therefore, we are going to ignore this constraint and assume that negative tips are acceptable.

(a) Suppose 6 people sit around a table and there are 6 plates of tips at the end.



If we know the amounts in every plate of tips (P_1 to P_6), can we determine the individual tips of all 6 people (T_1 to T_6)? If yes, explain why. If not, give two different assignments of T_1 to T_6 that will result in the same P_1 to P_6.

Solution:

No, this is not possible to determine in general. For example, the following two different assignments of tip amounts for each person:

$$(T_1, T_2, T_3, T_4, T_5, T_6) = (2, 0, 2, 0, 2, 0)$$
$$(T_1, T_2, T_3, T_4, T_5, T_6) = (0, 2, 0, 2, 0, 2)$$

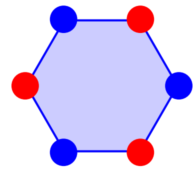
Will both result in $(P_1, P_2, P_3, P_4, P_5, P_6) = (1, 1, 1, 1, 1, 1)$.

If we write down the system of linear equations:

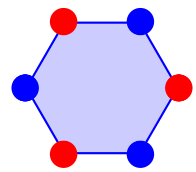
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 2P_1 \\ 2P_2 \\ 2P_3 \\ 2P_4 \\ 2P_5 \\ 2P_6 \end{bmatrix}$$

We can use Gaussian elimination to reduce the last row to all zeros. Therefore, equations are linearly dependent. However, Gaussian elimination is not needed to solve this question.

Intuitively, we can color each spot on the table alternating between red or blue:

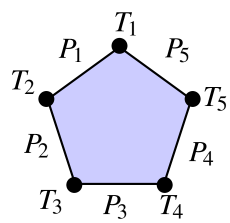


Then supposing that everyone sitting at red spots all tip r dollars and everyone sitting at blue spots all tip b dollars, we find $P_1, \cdots P_6$ dollars on the plates. However, this is no different from this following coloring:



Thus, we see that because of the special symmetry of the six-sided table, it's not possible to deduce everyone's tip.

(b) The same question as above, but what if we have 5 people sitting around a table?



Solution:

Yes. The problem can be reduced to the following system of equations:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 2P_1 \\ 2P_2 \\ 2P_3 \\ 2P_4 \\ 2P_5 \end{bmatrix}$$

We can then run row-reduction on this matrix. First subtract rows 1 and 3 and add rows 2 and 4 to row 5:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 2P_1 \\ 2P_2 \\ 2P_3 \\ 2P_4 \\ P_5 - P_1 + P_2 - P_3 + P_4 \end{bmatrix}$$

Now we subtract row 5 from row 4, row 4 from row 3, row 3 from row 2 and row 2 from row 1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} P_1 - P_2 + P_3 - P_4 + P_5 \\ P_2 - P_3 + P_4 - P_5 + P_1 \\ P_3 - P_4 + P_5 - P_1 + P_2 \\ P_4 - P_5 + P_1 - P_2 + P_3 \\ P_5 - P_1 + P_2 - P_3 + P_4 \end{bmatrix}$$

Thus, we have a unique solution for T_1 to T_5 in terms of P_1 to P_5.

Intuitively, unlike the argument for the previous part, since there is an odd number of seats, it's not possible for people to color every alternate seat red or blue.

(c) If n is the total number of people sitting around a table, for which n can you figure out everyone's tip? You do not have to rigorously prove your answer.

Solution:

Note: Although you didn't need to prove your answers rigorously, we will give you a rigorous argument. As long as your answer has the flavor of an argument (or an another equally sound argument), you should give yourself full credit.

For even n, this is not possible. Here is a counterexample: Suppose that all the plates had \$1 in them. There are clearly two different ways that this could have happened.

First:

$$T_n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Second:

$$T_n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

Both cases will result in $P_i = 1$ for all i from 1 to n. Therefore, we cannot figure out everyone's tip.

We can determine everyone's tips for all odd n. You can either argue with Gaussiam elimination on a general $n \times n$ matrix where n is odd or use the second argument, which does not use Gaussian elimination:

Gaussian elimination solution:

For odd n, the matrix encoding the system of linear equations is:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n-2 \\ \text{Row } n-1 \\ \text{Row } n \end{matrix}$$

We want to perform Gaussian elimination on this matrix. First, we subtract all odd-numbered rows from row n and add all even-numbered rows to row n. What is row n in the end? Denote the i-th item in row n by $R_{n,i}$. We know that $R_{n,1} = 0$, and for $i = 2, \cdots, n-2$, $R_{n,i} = 1 - 1 = 0$. Since n is odd, row $n-2$ is subtracted from row n, and row $n-1$ is added to row n. Therefore, $R_{n,n-1} = 0$ and $R_{n,n} = 2$. Now we can divide row n by 2, then subtract row n from row $n-1$, row $n-1$ from row $n-2$, and so on, until we get the identity matrix. Therefore we can see that all the rows are linearly independent and we can obtain a unique solution to this system of equations.

Alternate solution:

Suppose that each customer tipping: $T_1 = a_1, \cdots, T_n = a_n$ gives rise to the amount in plates $P_1 = p_1, \cdots, P_n = p_n$. Now suppose there exist another different solution to $T_1, \cdots, T_n$ that gives the same amount in plates. Then for some i two possible values for T_i, a_i and $a_i + \varepsilon$, are possible. Then:

$$\begin{aligned} a_i + a_{i+1} &= 2p_i \\ T_{i+1} &= 2p_i - T_i \\ &= (a_i + a_{i+1}) - (a_i + \varepsilon) \\ &= a_{i+1} - \varepsilon \end{aligned}$$

This means that $T_{i+1} = a_{i+1} - \varepsilon$, $T_{i+2} = a_{i+2} + \varepsilon$ and so on. We can then keep going around the circle, noting that there are an odd amount of sign flips in total before we get back to T_i. Therefore, we eventually get $T_i = a_i - \varepsilon$. However, we are assuming that $T_i = a_i + \varepsilon$ in the beginning. Therefore ε must be zero and we conclude that there is one unique solution.

6. Image Stitching

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options how they can capture the entire object:

- Stand as far as away as they need to to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object, and stitch them together, like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and rotation matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images and it's your job to figure out how to stitch the images together. You recently learned about vectors and rotation matrices in EE16A, and you have an idea about how to do this.

Your idea is that you should be able to find a single rotation matrix, $\mathbf{R}$, which is a function of some angle, θ, and a translation vector, $\vec{T}$, that transforms every common point in one image to that same point in the other image. Once you find the the angle, θ, and the translation vector, $\vec{T}$, you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p}$ is a point in one image and $\vec{q}$ is the corresponding point (i.e., they represent the same thing in the scene) in the other image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}(\theta)} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

This looks good, but then you realize that one of the pictures might be farther away than the other. You realize that you need to add a scaling factor, $\lambda > 0$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \lambda \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \tag{5}$$

(For example, if $\lambda > 1$, then the image containing q is closer (appears larger) than the image containing p. If $0 < \lambda < 1$, then the image containing q appears smaller.)

You are now confident that if you can find θ, $\vec{T}$, and λ, you will be able to reorient and scale one of the images, so that it lines up with the other image.

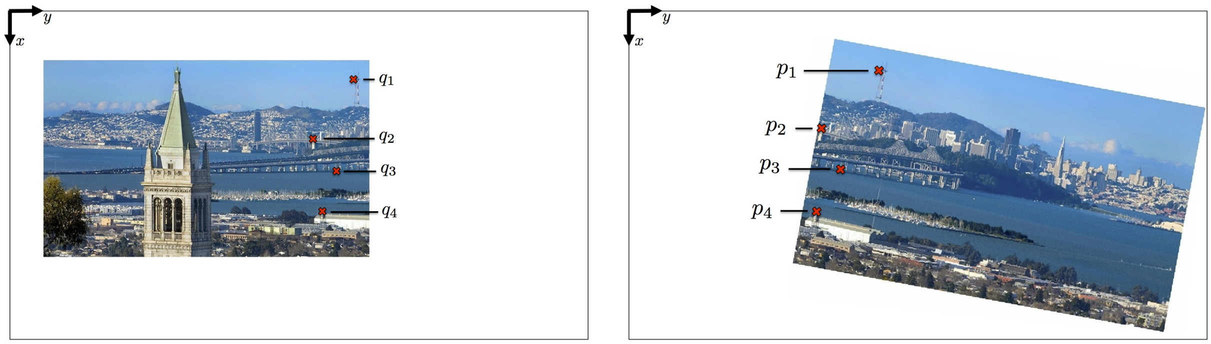


Figure 3: Two images to be stitched together with pairs of matching points labeled.

Before you get too excited, however, you realize that you have a problem. Equation 5 is not a linear equation with respect to θ, $\vec{T}$, and λ. You're worried that you don't have a good technique for solving nonlinear systems of equations. You decide to talk to Marcela and the two of you come up with a brilliant solution.

You decide to "relax" the problem, so that you're solving for a general matrix $\mathbf{R}$ rather than a perfect scaled rotation matrix. The new equation you come up with is:

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \tag{6}$$

This equation is linear, so you can solve for $R_{xx}, R_{xy}, R_{yx}, R_{yy}, T_x$, and T_y. Also you realize that if $\vec{p}$ and $\vec{q}$ actually do differ by a rotation of θ degrees and a scaling of λ, you can expect that the general matrix $\mathbf{R}$ that you find will turn out to be a scaled rotation matrix with $R_{xx} = \lambda\cos(\theta)$, $R_{xy} = -\lambda\sin(\theta)$, $R_{yx} = \lambda\sin(\theta)$, and $R_{yy} = \lambda\cos(\theta)$.

(a) Multiply Equation 6 out into two scalar linear equations. What are the known values and what are the unknowns in each equation? How many unknowns are there? How many equations do you need to solve for all the unknowns? How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns?

Solution:

We can rewrite the above matrix equation as the following two scalar linear equations:

$$q_x = p_x R_{xx} + p_y R_{xy} + T_x$$
$$q_y = p_x R_{yx} + p_y R_{yy} + T_y$$

Here, the known values are the elements of the pair of points: q_x, q_y, p_x, p_y, and 1. The unknowns are elements of $\mathbf{R}$ and $\vec{T}$: R_{xx}, R_{xy}, R_{yx}, R_{yy}, T_x, and T_y. There are 6 unknowns, so we need a total of 6 equations to solve for them. For every pair of points we add, we get two more equations. Thus, we need 3 pairs of common points to get 6 equations.

(b) Write out a system of linear equations that you can use to solve for the values of $\mathbf{R}$ and $\vec{T}$.

Solution:

We will label the 3 pairs of point we select as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}$$

We write the system of linear equations in matrix form.

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{xx} \\ R_{xy} \\ R_{yx} \\ R_{yy} \\ T_x \\ T_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

(c) In the IPython notebook `prob2.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the parameters $\mathbf{R}$ and $\vec{T}$. You will be prompted to enter your results, and the notebook will then apply your transformation to the second image and show you if your stitching algorithm works.

Solution:

The parameters for the transformation from the coordinates of the first image to those of the second image are $\mathbf{R} = \begin{bmatrix} 1.1954 & .1046 \\ -.1046 & 1.1954 \end{bmatrix}$ (this corresponds to $\lambda = 1.2$, $\theta = -5°$), and $\vec{T} = \begin{bmatrix} -150 \\ -250 \end{bmatrix}$.

(d) We will now explore when this algorithm fails. For example, the three pairs of points must all be distinct points. Show that if $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are *collinear*, the system of equations (6) is underdetermined. Does this make sense geometrically?
(Think about the kinds of transformations possible by a general affine transformation. An affine transformation is one that preserves points. For example, in the rotation of a line, the angle of the line might change, but the length will not. All linear transformations are affine. **Definition of Affine.**)
Use the following fact: $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are collinear iff $(\vec{p}_2 - \vec{p}_1) = k(\vec{p}_3 - \vec{p}_1)$ for some $k \in \mathbb{R}$.

Solution:

Note: A general 2D affine transformation is a transformation from points $\vec{p}$ to points $\vec{q}$ of the form of Equation 6. It is a linear transformation (given by matrix $\mathbf{R}$) followed by a translation given by the vector $\vec{T}$.

The algorithm fails when the points $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are not distinct or when they are collinear.

To show that the collinear case leads to an underdetermined system, write the system (with respect to the matrix $\mathbf{R}$) as:

$$\begin{cases} \vec{q}_1 = \mathbf{R}\vec{p}_1 + \vec{T} \\ \vec{q}_2 = \mathbf{R}\vec{p}_2 + \vec{T} \\ \vec{q}_3 = \mathbf{R}\vec{p}_3 + \vec{T} \end{cases}$$

By subtracting the first equation from the last two, this is equivalent to the following system:

$$\begin{cases} \vec{q}_1 = \mathbf{R}\vec{p}_1 + \vec{T} \\ \vec{q}_2 - \vec{q}_1 = \mathbf{R}(\vec{p}_2 - \vec{p}_1) \\ \vec{q}_3 - \vec{q}_1 = \mathbf{R}(\vec{p}_3 - \vec{p}_1) \end{cases}$$

Then, by collinearity, this is equivalent to the system:

$$\begin{cases} \vec{q}_1 = \mathbf{R}\vec{p}_1 + \vec{T} \\ \vec{q}_2 - \vec{q}_1 = \mathbf{R}k(\vec{p}_3 - \vec{p}_1) \\ \vec{q}_3 - \vec{q}_1 = \mathbf{R}(\vec{p}_3 - \vec{p}_1) \end{cases}$$

Note that the last two equations are now linearly dependent. To see this, remember that the variables are the entries of the matrix $\mathbf{R} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}$, and all other elements $(\vec{p}_i, \vec{q}_i, k)$ are constants. Define $\vec{p}_{31} = \vec{p}_3 - \vec{p}_1$. The third equation is a constraint on linear combinations of entries in $\mathbf{R}$ – specifically, it is a constraint on $\mathbf{R}\vec{p}_{31}$. The second equation is a constraint on $\mathbf{R}(k\vec{p}_{31})$. However, since $\mathbf{R}(k\vec{p}_{31}) = k(\mathbf{R}\vec{p}_{31})$, these two constraints are linearly dependent. Therefore the system is underdetermined.

This problem can also be solved by writing out the linear equations explicitly in matrix form, as you did in part (b). The system becomes:

$$\underbrace{\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} R_{xx} \\ R_{xy} \\ R_{yx} \\ R_{yy} \\ T_x \\ T_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

Subtracting row 1 from rows 4 and 5 and then subtracting row 2 from rows 4 and 6, the matrix $\mathbf{A}$ becomes:

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} - p_{1x} & p_{2y} - p_{1y} & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{2x} - p_{1x} & p_{2y} - p_{1y} & 0 & 0 \\ p_{3x} - p_{1x} & p_{3y} - p_{1y} & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{3x} - p_{1x} & p_{3y} - p_{1y} & 0 & 0 \end{bmatrix}$$

Then, by collinearity, we have $p_{2x} - p_{1x} = k(p_{3x} - p_{1x})$ and so on, so the above matrix is:

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ k(p_{3x} - p_{1x}) & k(p_{3y} - p_{1y}) & 0 & 0 & 0 & 0 \\ 0 & 0 & k(p_{3x} - p_{1x}) & k(p_{3y} - p_{1y}) & 0 & 0 \\ p_{3x} - p_{1x} & p_{3y} - p_{1y} & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{3x} - p_{1x} & p_{3y} - p_{1y} & 0 & 0 \end{bmatrix}$$

Now, clearly rows 3 and 5 are linearly dependent (and, in fact, so are rows 4 and 6). Therefore the linear system is underdetermined.

Geometrically: Consider a transformation that simply scales a vector along the x-axis. For example, $\vec{T} = 0$ and $\mathbf{R} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Say we pick all $\vec{p}_i$ to lie along the y-axis. Then, applying this transformation will give $\vec{q}_i = \vec{p}_i$ (since the y-axis is unaffected by this scaling). However, only given q_i, this transformation has the same effect as the identity transformation (that is, where $\vec{T} = \vec{0}$ and $\mathbf{R}$ is the identity). Therefore, the transformation cannot be determined from the q_i (both the identity transformation and the x-axis scaling lead to the same q_i). This argument can be extended to the case when all p_i are collinear but not necessarily along the x-axis.

(e) **(PRACTICE)** Show that if the three points are not collinear, the system is fully determined.

Solution:

As before, write the system (with respect to the matrix $\mathbf{R}$) as:

$$\begin{cases} \vec{q}_1 = \mathbf{R}\vec{p}_1 + \vec{T} \\ \vec{q}_2 = \mathbf{R}\vec{p}_2 + \vec{T} \\ \vec{q}_3 = \mathbf{R}\vec{p}_3 + \vec{T} \end{cases}$$

We can subtract the first equation from the last two to get:

$$\begin{cases} \vec{q}_2 - \vec{q}_1 = \mathbf{R}(\vec{p}_2 - \vec{p}_1) \\ \vec{q}_3 - \vec{q}_1 = \mathbf{R}(\vec{p}_3 - \vec{p}_1) \end{cases}$$

Define $\vec{p}_{21} = \vec{p}_2 - \vec{p}_1$ and $\vec{p}_{31} = \vec{p}_3 - \vec{p}_1$ and similarly $\vec{q}_{21} = \vec{q}_2 - \vec{q}_1$ and $\vec{q}_{31} = \vec{q}_3 - \vec{q}_1$.
Then the above system can be written as:

$$\begin{bmatrix} \vec{q}_{21} & \vec{q}_{31} \end{bmatrix} = \mathbf{R} \begin{bmatrix} \vec{p}_{21} & \vec{p}_{31} \end{bmatrix}$$

Or, writing out $\mathbf{R}$ explicitly:

$$\begin{bmatrix} \vec{q}_{21} & \vec{q}_{31} \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \vec{p}_{21} & \vec{p}_{31} \end{bmatrix}$$

This can be written as a system of two equations, one for each row of the matrix $\mathbf{R}$. For example,

$$\vec{\alpha}_1^T = \begin{bmatrix} R_{xx} & R_{xy} \end{bmatrix} \begin{bmatrix} \vec{p}_{21} & \vec{p}_{31} \end{bmatrix},$$

where $\vec{\alpha}_1^T$ is the first row of matrix the $\begin{bmatrix} \vec{q}_{21} & \vec{q}_{31} \end{bmatrix}$.
Then, transposing the equation gives:

$$\begin{bmatrix} \vec{p}_{21} & \vec{p}_{31} \end{bmatrix}^T \begin{bmatrix} R_{xx} \\ R_{xy} \end{bmatrix} = \vec{\alpha}_1$$

Notice that by non-collinearity, the vectors $\vec{p}_{21}$ and $\vec{p}_{31}$ are not linearly dependent. Therefore, the 2×2 matrix $\begin{bmatrix} \vec{p}_{21} & \vec{p}_{31} \end{bmatrix}^T$ has linearly independent rows, so this system can be solved for R_{xx} and R_{xy}, the first row of $\mathbf{R}$. The second row of $\mathbf{R}$ can be found analogously.
Then, we can find $\vec{T}$, by using one of the correspondence pairs $\vec{T} = \vec{q}_1 - \mathbf{R}\vec{p}_1$.

(f) **(PRACTICE)** Marcela comments that perhaps the system (with three collinear points) is only under-determined because we "relaxed" our model too much by allowing for general affine transformations, instead of just isotropic-scale/rotation/translation. Can you come up with a different representation of Equation 5, that will allow for recovering the transform from only *two* pairs of distinct points? *(Hint: Let $a = \lambda\cos(\theta)$ and $b = \lambda\sin(\theta)$. In other words, enforce $R_{xx} = R_{yy}$ and $R_{xy} = -R_{yx}$).*

Solution:

Model the system with respect to the two variables $a, b, \in \mathbb{R}$ as:

$$\vec{q} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{p} + \vec{T}$$

Given two pairs of points p_i, q_i, we have the system:

$$\begin{cases} \vec{q}_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{p}_1 + \vec{T} \\ \vec{q}_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{p}_2 + \vec{T} \end{cases}$$

We can subtract the equations to get:

$$\vec{q}_2 - \vec{q}_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} (\vec{p}_2 - \vec{p}_1)$$

Define $\vec{v} = \vec{q}_2 - \vec{q}_1$ and $\vec{w} = \vec{p}_2 - \vec{p}_1$. Then, writing the above system in components:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

This can be written explicitly as a linear system with respect to the variables a, b as:

$$\underbrace{\begin{bmatrix} w_x & -w_y \\ w_y & w_x \end{bmatrix}}_{\mathbf{W}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

If $\vec{p}_1 \neq \vec{p}_2$, then $\vec{w} \neq \vec{0}$, and the matrix $\mathbf{W}$ has linearly independent rows and columns. Therefore, we can solve for a, b given only two pairs of distinct points.

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.