EECS 16A Designing Information Devices and Systems I
Spring 2018 Homework 3

This homework is due February 12, 2018, at 23:59.
Self-grades are due February 15, 2018, at 23:59.

Submission Format

Your homework submission should consist of **two** files.

- `hw3.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible.

- `hw3.ipynb`: A single IPython notebook with all of your code in it.

  In order to receive credit for your IPython notebook, you must submit both a "printout" and the code itself.

Submit each file to its respective assignment on Gradescope.

1. Mechanical Inverses

For each of the following matrices, state whether the inverse exists. If so, find the inverse. If not, show why no inverse exists. **Solve these by hand. Do NOT use IPython for this problem.**

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right]$$

Inverse exists: $\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

(b) **PRACTICE:** $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 5 & 0 & 1 & 2 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} \end{array}\right]$$

Inverse exists: $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}$

(c) **PRACTICE:** $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & -2 & \sqrt{3} & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \sqrt{3} & -2 & 0 \\ 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array}\right]$$

Inverse exists: $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array}\right]$$

Inverse exists: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Inverse exists: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

Solution:

Inverse does not exist because the second and third column are equal, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(g) **PRACTICE:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution:

Inverse does not exist because the third column is the negative of the second column, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(h) **PRACTICE:** $\begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 1 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 1 & 1 & -2 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 1 & 1 & -2 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 1 & 1 & -2 \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Inverse exists: $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(i) $\begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Solution:

Inverse does not exist because $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{v}_4$. Since the columns of the matrix are linearly dependent, the inverse does not exist.

2. Pumps Properties Proofs

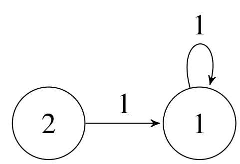


Figure 1: Pump system.

(a) Suppose that we have a pump setup as in Figure 1 with the associated transition matrix $\mathbf{A}$. Write out the state transition matrix $\mathbf{A}$.

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) Suppose that the reservoirs are initialized to the following water levels: $x_1[0] = 0.5, x_2[0] = 0.5$. In a completely alternate universe, the reservoirs are initialized to the following water levels: $x_1[0] = 0.3, x_2[0] = 0.7$. For both initial states, what are the water levels at timestep 1 ($\vec{x}[1]$)?

Solution:

$$\vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$$

$$\vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) If you observe the reservoirs at timestep 1, can you figure out what the initial ($\vec{x}[0]$) water levels were? Why or why not?

Solution:

No, at timestep 1, configuration 1 and 2 are indistinguishable. This implies that the system is inherently linearly dependent.

(d) Now generalize: if there exists a state transition matrix where two different initial state vectors lead to the same water levels/state vectors at a timestep in the future, can you recover the initial water levels? Prove your answer.

(*Hint: What does this say about the matrix* $\mathbf{A}$*?*)

Solution:

We are told that two different initial states, $\vec{x}[0]$ and $\vec{y}[0]$, lead to the same resulting state $\vec{x}[1]$.

$$\mathbf{A}\vec{x}[0] = \vec{x}[1] \qquad \mathbf{A}\vec{y}[0] = \vec{x}[1]$$

If we can recover the initial water levels, this would mean that there is some operation that can be performed on the resulting state vector to yield the previous state. This operation would, by definition, be the inverse of the matrix $\mathbf{A}$.

$$\vec{x}[0] = \mathbf{A}^{-1}\vec{x}[1] \qquad \vec{y}[0] = \mathbf{A}^{-1}\vec{x}[1]$$

However, we can see that the above two equations contradict each other. Therefore, $\mathbf{A}$ is not invertible, and you cannot recover the previous states from the resulting state.

(e) Suppose that there is a state transition matrix, where every initial state is guaranteed to have a unique state vector in the next timestep. Consider what this statement implies about the system of linear equations represented by the state transition matrix $\mathbf{A}$. Can you recover the initial state? Prove your answer, and explain what this implies about the system.

Solution:

Since each unique input leads to a unique output, we know that the state transition matrix $\mathbf{A}$ is full rank. This means that the matrix is invertible. This implies that, for this system, we can determine the previous states from the current state.

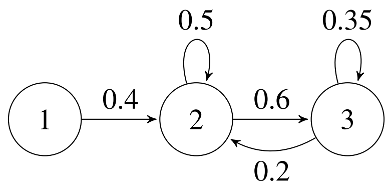
(f) Suppose that there is a state transition matrix such that the entries of each column vector sum to one. What is the physical interpretation about the total amount of water in the system?

Solution:

The total amount of water in the system remains the same. There are no “leaks/drains” or “inlets/gains” in the system.

(g) **(PRACTICE)** Set up the state transition matrix $\mathbf{A}$ for the system of pumps shown below. Explain what this $\mathbf{A}$ matrix physically implies about the total amount of water in this system.

Note: If there is no "self-arrow/self-loop," then the water does not return.



Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0.4 & 0.5 & 0.2 \\ 0 & 0.6 & 0.35 \end{bmatrix}$$

Note that the entries in the columns do *not* sum to one. This is physically interpreted as a "leak" – i.e., the total amount of water is not conserved.

(h) **(PRACTICE)** There is a state transition matrix where the entries of its rows sum to one. Prove that applying this system to a uniform vector will return the same uniform vector. A uniform vector is a vector whose elements are all the same.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}$$

Solution:

Consider the row interpretation of matrix multiplication: Each b_i is equal to the dot product of the row vector $\vec{\alpha}_i^T$ and the vector $\vec{x}$.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_n^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

For example, let's look at the first row b_1.

$$b_1 = \vec{\alpha}_1^T \vec{x} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

All values in the vector $\vec{x}$ are equal to x.

$$\begin{aligned} b_1 &= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{bmatrix} \begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix} \\ &= a_{11}x + a_{12}x + \cdots + a_{1n}x \\ &= x(a_{11} + a_{12} + \cdots + a_{1n}) \end{aligned}$$

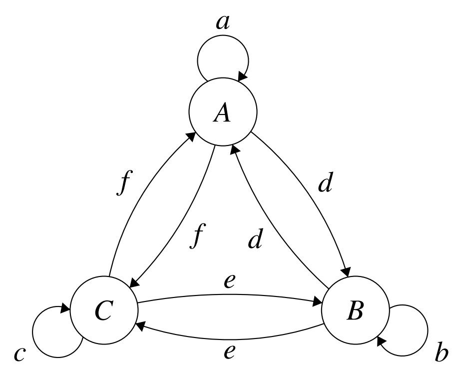
Remember that the values in $\vec{\alpha}_1^T$ sum to 1.

$$b_1 = x(a_{11} + a_{12} + \cdots + a_{1n}) = x \cdot 1 = x$$

As we can see, $b_1 = x$. This property carries through for each row, so all of them sum to 1. Thus $\vec{b} = \vec{x}$, which is what we wanted to prove.

3. **Reservoirs That Give and Take**

Consider a network of three water reservoirs A, B, and C. At the end of each day, water transfers among the reservoirs according to the directed graph shown below.



The parameters a, b, and c—which label the self-loops—denote the *fractions* of the water in reservoirs A, B, and C, respectively, that stay in the same reservoir at the end of each day n. The parameters d, e, and f denote the fractions of the reservoir contents that transfer to adjacent reservoirs at the end of each day, according to the directed graph above.

Assume that the reservoir system is conservative—no water enters or leaves the system, which means that the total water in the network is constant. Accordingly, *for each node*, the weights on its self-loop and its two outgoing edges sum to 1; for example, for node A, we have

$$a + d + f = 1,$$

and similar equations hold for the other nodes. Moreover, assume that all the edge weights are positive numbers—that is,

$$0 < a, b, c, d, e, f < 1.$$

The state evolution equation governing the water flow dynamics in the reservoir system is given by $\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n]$, where the 3×3 matrix $\mathbf{A}$ is the state transition matrix, and $\vec{s}[n] = \begin{bmatrix} s_A[n] & s_B[n] & s_C[n] \end{bmatrix}^T \in \mathbb{R}^3$ is the nonnegative state vector that shows the water distribution among the three reservoirs at the end of day n, as fractions of the total water in the network.

(a) Determine the state transition matrix $\mathbf{A}$.

Solution:

$$\mathbf{A} = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}$$

(b) For some systems, there exists an equilibrium state —that is, a state for which the following is true:

$$\vec{s}[n+1] = \vec{s}[n] = \vec{s}^*$$

Determine if for the systems described above, there exists a equilibrium state. If such a state exists, find it.

Hint: What's special about the rows of this matrix?

Solution:

Notice for the above matrix, $\mathbf{A}^T = \mathbf{A}$. Such a matrix is called symmetric. This implies both the rows and the columns of this matrix sum to one.

Consider what happens when you apply this matrix to a uniform vector, $\vec{s}[0] = [x\ x\ x]^T = x[1\ 1\ 1]^T$.

$$\vec{s}[1] = \mathbf{A}\vec{s}[0] = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix} \begin{bmatrix} x \\ x \\ x \end{bmatrix} = \begin{bmatrix} ax+dx+fx \\ dx+bx+ex \\ fx+ex+cx \end{bmatrix} = \begin{bmatrix} x(a+d+f) \\ x(d+b+e) \\ x(f+e+c) \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

Since $\vec{s}[1] = \vec{s}[0]$, $\vec{s}$ is the equilibrium state $\vec{s}^*$.

An alternative way of doing the problem would be to compute the nullspace of $\mathbf{A} - \mathbf{I}$ in the following way.

$$\begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow (\mathbf{A} - \mathbf{I})\,[x_1 x_2 x_3]^T = \mathbf{0}$$

$$\begin{bmatrix} a-1 & d & f \\ d & b-1 & e \\ f & e & c-1 \end{bmatrix} \sim \begin{bmatrix} a-1 & d & f \\ d & b-1 & e \\ a+d+f-1 & b+d+e-1 & e+f+c-1 \end{bmatrix} \sim \begin{bmatrix} a-1 & d & f \\ d & b-1 & e \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we can set x_3 as a free variable (say $x_3 = 1$), which gives us the following 2 equations for x_1, x_2

$$(a-1)x_1 + dx_2 + f = 0 = (-d-f)x_1 + dx_2 + f$$

$$dx_2 + (b-1)x_2 + e = 0 = dx_1 + (-d-e)x_2 + e$$

Adding the above two equations gives

$$f(1-x_1) + e(1-x_2) = 0$$

Thus

$$x_1 = x_2 = x_3 = 1$$

is a basis for the nullspace, and $[x\ x\ x]^T$ is a general solution.

(c) Suppose the state transition matrix for the network is given by

$$\mathbf{A} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}.$$

Is it possible to determine the state $\vec{s}[n]$ from the subsequent state $\vec{s}[n+1]$? You do not have to find $\vec{s}[n]$ from $\vec{s}[n+1]$, just determine whether it is possible to do so.

Solution:

We row-reduce $\mathbf{A}$ and see what the pivots look like. If they're all nonzero, then $\mathbf{A}$ is invertible and reverse-time inference (obtaining $\vec{s}[n]$ from $\vec{s}[n+1]$) is possible. If even a single pivot is zero, then $\mathbf{A}$ is not invertible, in which case we cannot determine $\vec{s}[n]$ from $\vec{s}[n+1]$.

$$\begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} \sim \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & -\frac{3}{5} & -\frac{2}{5} \\ 0 & -\frac{2}{5} & -\frac{2}{5} \end{bmatrix} \sim \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & -\frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & -\frac{2}{5} \end{bmatrix}$$

All three pivots are nonzero. Therefore $\mathbf{A}$ is invertible and $\vec{s}[n] = \mathbf{A}^{-1}\vec{s}[n+1]$. The problem does not ask us to compute $\mathbf{A}^{-1}$ but merely whether it's possible to determine $\vec{s}[n]$ from $\vec{s}[n+1]$. The answer is yes!

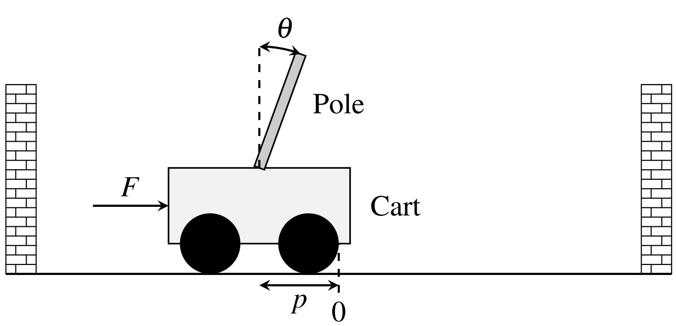
4. Bieber's Segway

After one too many infractions with the law, J-Biebs decides to find a new mode of transportation, and you suggest he get a segway.

He becomes intrigued by your idea and asks you how it works.

You let him know that a force (through the spinning wheel) is applied to the base of the segway, and this in turn controls both the position of the segway and the angle of the stand. As you push on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

J-Biebs is impressed, to say the least, but he is a little concerned that only one input (force) is used to control two outputs (position and angle). He finally asks if it's possible for the segway to be brought upright and to a stop from any initial configuration. J-Biebs calls up a friend who's majoring in mechanical engineering, who tells him that a segway can be modeled as a cart-pole system:



A cart-pole system can be fully described by its position p, velocity $\dot{p}$, angle θ, and angular velocity $\dot{\theta}$. We write this as a "state vector":

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

The input to this system u will just be the force applied to the cart (or base of the segway).[1]

At time step n, we can apply scalar input $u[n]$. The cart-pole system can be represented by a linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n], \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{4\times4}$ and $\vec{b} \in \mathbb{R}^{4\times1}$. The model tells us how the the state vector will evolve over (discrete) time as a function of the current state vector and control inputs.

To answer J-Biebs' question, you look at this general linear system and try to answer the following question: Starting from some initial state $\vec{x}_0$, can we reach a final desired state, $\vec{x}_f$, in N steps?

The challenge seems to be that the state is 4-dimensional and keeps evolving and that we can only apply a one dimensional control at each time. Is it possible to control something 4-dimensional with only one degree of freedom that we can control?

You will solve this problem by walking through several steps.

(a) Express $\vec{x}[1]$ in terms of $\vec{x}[0]$ and the input $u[0]$.

Solution:

From Equation 1, we get (by substituting $n = 0$):

$$\vec{x}[1] = \mathbf{A}\vec{x}[0] + \vec{b}u[0] \tag{2}$$

(b) Express $\vec{x}[2]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$ and $u[1]$. Then express $\vec{x}[3]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, and $u[2]$, and express $\vec{x}[4]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, $u[2]$, and $u[3]$.

Solution:

From Equation 1, we get (by substituting $n = 1$):

$$\vec{x}[2] = \mathbf{A}\vec{x}[1] + \vec{b}u[1]$$

By substituting $\vec{x}[1]$ from Equation 2, we get

$$\begin{aligned}\vec{x}[2] &= \mathbf{A}\vec{x}[1] + \vec{b}u[1] \\ &= \mathbf{A}\left(\mathbf{A}\vec{x}[0] + \vec{b}u[0]\right) + \vec{b}u[1] \\ &= \mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1] \end{aligned} \tag{3}$$

From Equation 1, we get (by substituting $n = 2$):

$$\vec{x}[3] = \mathbf{A}\vec{x}[2] + \vec{b}u[2]$$

[1] Some of you might be wondering why it is that applying a force in this model immediately causes a change in position. You might have been taught in high school physics that force creates acceleration, which changes velocity (both simple and angular), which in turn causes changes to position and angle. Indeed, when viewed in continuous time, this is true instantaneously. However, here in this engineering system, we have discretized time, i.e. we think about applying this force constantly for a given finite duration and we see how the system responds after that finite duration. In this finite time, indeed the application of a force will cause changes to all four components of the state. But notice that the biggest influence is indeed on the two velocities, as your intuition from high school physics would predict.

By substituting $\vec{x}[2]$ from Equation 3, we get

$$\begin{aligned}\vec{x}[3] &= \mathbf{A}\vec{x}[2] + \vec{b}u[2] \\ &= \mathbf{A}\left(\mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1]\right) + \vec{b}u[1] \\ &= \mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2] \end{aligned} \tag{4}$$

From Equation 1, we get (by substituting $n = 3$):

$$\vec{x}[4] = \mathbf{A}\vec{x}[3] + \vec{b}u[3]$$

By substituting $\vec{x}[3]$ from Equation 4, we get

$$\begin{aligned}\vec{x}[4] &= \mathbf{A}\vec{x}[3] + \vec{b}u[3] \\ &= \mathbf{A}\left(\mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2]\right) + \vec{b}u[3] \\ &= \mathbf{A}^4\vec{x}[0] + \mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] \end{aligned} \tag{5}$$

(c) Now, derive an expression for $\vec{x}[N]$ in terms of $\vec{x}[0]$ and the inputs from $u[0],\ldots,u[N-1]$. (*Hint: You can use a summation from* 0 *to* $N-1$.)

Solution:

Use the same procedure as above for N steps. You will obtain the following expression:

$$\vec{x}[N] = \mathbf{A}^N\vec{x}[0] + \sum_{i=0}^{N-1}\mathbf{A}^i\vec{b}u[N-i-1] \tag{6}$$

Note that $\mathbf{A}^0$ is the identity matrix.

As a sanity check, plug the values $N = 1,2,3,$ and 4 to obtain Equations 2, 3,4, and 5, respectively.

For the next four parts of the problem, you are given the matrix $\mathbf{A}$ and the vector $\vec{b}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

The state vector $\vec{0}$ corresponds to the cart-pole (or segway) being upright and stopped at the origin. Assume the cart-pole starts in an initial state $\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$, and you want to reach the desired state $\vec{x}_f = \vec{0}$ using the control inputs $u[0], u[1], \ldots$. (Reaching $\vec{x}_f = \vec{0}$ in N steps means that, given that we start at $\vec{x}[0]$, we can find control inputs, such that we get $\vec{x}[N]$, the state vector at the Nth time step, equal to $\vec{x}_f$.)

Note: You may use IPython to solve the next four parts of the problem. We have provided a function `gauss_elim(matrix)` to help you find the reduced row echelon form of matrices.

(d) Can you reach $\vec{x}_f$ in *two* time steps? (*Hint: Express* $\vec{x}[2] - \boldsymbol{A}^2\vec{x}[0]$ *in terms of the inputs* $u[0]$ *and* $u[1]$.)

Solution:

No.

From Equation 3, we know that $\mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}[2]$ which is equivalent to $\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}[2] - \mathbf{A}^2\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in two time steps (that is, $\vec{x}[2] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}_f - \mathbf{A}^2\vec{x}[0],$$

where $u[0]$ and $u[1]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = -\mathbf{A}^2\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | \\ \mathbf{A}\vec{b} & \vec{b} \\ | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix} = -\mathbf{A}^2\vec{x}[0],$$

which yields the following augmented matrix:

$$\left[\begin{array}{cc|c} | & | & | \\ \mathbf{A}\vec{b} & \vec{b} & -\mathbf{A}^2\vec{x}[0] \\ | & | & | \end{array}\right].$$

By plugging in the values of $\mathbf{A}$, $\vec{b}$, and $\vec{x}[0]$, we get the following augmented matrix:

$$\left[\begin{array}{cc|c} 0.0208 & 0.01 & 0.02243475295 \\ 0.0557 & 0.21 & -0.30785116611 \\ -0.0572 & -0.03 & 0.0619347608 \\ -0.2385 & -0.44 & 1.38671325508 \end{array}\right].$$

Applying Gaussian elimination, we get the reduced row echelon form to be

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right],$$

which means that the system is inconsistent (due to the third row) and that there are no solutions for $u[0]$ and $u[1]$. It is fine if you did not row reduce all the way to the reduced row echelon form as long as you showed that the system of equations is inconsistent.

(e) Can you reach $\vec{x}_f$ in *three* time steps?

Solution:

No.

Similar to the previous part, from Equation 4, we know that $\mathbf{A}^3\vec{x}[0]+\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=\vec{x}[3]$, which is equivalent to $\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=\vec{x}[3]-\mathbf{A}^3\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in three time steps (that is, $\vec{x}[3]=\vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=\vec{x}_f-\mathbf{A}^3\vec{x}[0],$$

where $u[0],u[1]$, and $u[2]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=-\mathbf{A}^3\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix}| & | & | \\ \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & |\end{bmatrix}\begin{bmatrix}u[0]\\u[1]\\u[2]\end{bmatrix}=-\mathbf{A}^3\vec{x}[0],$$

which yields the following augmented matrix:

$$\left[\begin{array}{ccc|c}| & | & | & | \\ \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} & -\mathbf{A}^3\vec{x}[0] \\ | & | & | & |\end{array}\right].$$

By plugging in the values of $\mathbf{A}$, $\vec{b}$, and $\vec{x}[0]$, we get the following augmented matrix:

$$\left[\begin{array}{ccc|c}0.024157 & 0.0208 & 0.01 & 0.0064228470365 \\ 0.024363 & 0.0557 & 0.21 & -0.092123298431 \\ -0.083488 & -0.0572 & -0.03 & 0.178491836209001 \\ -0.342448 & -0.2385 & -0.44 & 1.246334243328597\end{array}\right].$$

Applying Gaussian elimination, we get the reduced row echelon form to be

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right],$$

which means that the system is inconsistent (due to the fourth row) and that there are no solutions for $u[0],u[1]$, and $u[2]$. It is fine if you did not row reduce all the way to the reduced row echelon form as long as you showed that the system of equations is inconsistent.

(f) Can you reach $\vec{x}_f$ in *four* time steps?

Solution:

Yes.

Similar to the previous part, from Equation 5, we know that $\mathbf{A}^4\vec{x}[0] + \mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}[4]$ which is equivalent to $\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}[4] - \mathbf{A}^4\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in four time steps (that is, $\vec{x}[4] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}_f - \mathbf{A}^4\vec{x}[0],$$

where $u[0], u[1], u[2]$, and $u[3]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = -\mathbf{A}^4\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | & | & | \\ \mathbf{A}^3\vec{b} & \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = -\mathbf{A}^4\vec{x}[0].$$

By defining $\mathbf{Q} = \begin{bmatrix} | & | & | & | \\ \mathbf{A}^3\vec{b} & \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & | & | \end{bmatrix}$ and $\vec{u}_4 = \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$, we can now rewrite our system of linear equations as

$$\mathbf{Q}\vec{u}_4 = -\mathbf{A}^4\vec{x}[0].$$

Refer to the code in the solution IPython notebook for a solution of the system above. The solution of the system is

$$\vec{u}_4 = \begin{bmatrix} -13.24875075 \\ 23.73325125 \\ -11.57181872 \\ 1.46515973 \end{bmatrix},$$

that is, the control input sequence is: $u[0] = -13.24875075$, $u[1] = 23.73325125$, $u[2] = -11.57181872$, and $u[3] = 1.46515973$.

(g) If the answer to the previous part is yes, find the required correct control inputs using IPython and verify the answer by entering these control inputs into the code in the IPython notebook. The code has been written to simulate this system, and you should see the system come to a halt in four time steps! *Suggestion: See what happens if you enter all four control inputs equal to 0. This gives you an idea of how the system naturally evolves!*

Solution:

See the solution to the previous part.

(h) Let's return to a general matrix $\mathbf{A}$ and a general vector $\vec{b}$. What condition do we need on

$$\text{span}\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$$

for $\vec{x}_f = \vec{0}$ to be "reachable" from $\vec{x}_0$ in N steps?

Solution:

Similar to the previous parts, the key step here is to rewrite the equation you derived in part (c) (Equation 6) as

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = \vec{x}[N] - \mathbf{A}^N x[0].$$

We want $\vec{x}[N] = \vec{x}_f = \vec{0}$. Therefore, the system of linear equations simplifies to

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = -\mathbf{A}^N x[0].$$

If we extend this sum, we get

$$\mathbf{A}^{N-1}\vec{b}u[0] + \mathbf{A}^{N-2}\vec{b}u[1] + \cdots + \mathbf{A}\vec{b}u[N-2] + \vec{b}u[N-1] = -\mathbf{A}^N\vec{x}[0].$$

This system of linear equations can be rewritten as

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix} = -\mathbf{A}^N\vec{x}[0].$$

We need to find $\{u[0], u[1], \ldots, u[N-1]\}$ that satisfy this system of linear equations. For this system to be solvable, we need $-\mathbf{A}^N x[0] \in \text{span}\left\{\vec{b}, A\vec{b}, A^2\vec{b}, \ldots, A^{N-1}\vec{b}\right\}$. That is, we need $-\mathbf{A}^N x[0]$ to be in the range (column space) of the matrix $\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}$.

(i) What condition would we need on $\text{span}\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$ for *any* valid state vector to be reachable from $\vec{x}_0$ in N steps?

Wouldn't this be cool?

Solution:

Similar to the previous parts, the key step here is to rewrite the equation you derived in part (c) (Equation 6) as

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = \vec{x}[N] - \mathbf{A}^N x[0].$$

The difference is that $\vec{x}[N] = \vec{x}_f$ can be anything in $\mathbb{R}^4$. Therefore, the system of linear equations can be written as

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = \vec{x}_f - \mathbf{A}^N x[0].$$

If we extend this sum, we get

$$\mathbf{A}^{N-1}\vec{b}u[0]+\mathbf{A}^{N-2}\vec{b}u[1]+\cdots+\mathbf{A}\vec{b}u[N-2]+\vec{b}u[N-1]=\vec{x}_f-\mathbf{A}^N x[0].$$

This system of linear equations can be further rewritten as

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}\begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix}=\vec{x}_f-\mathbf{A}^N x[0].$$

For this system to be solvable, we need $\vec{x}_f-\mathbf{A}^N x[0]\in\text{span}\left\{\vec{b},\mathbf{A}\vec{b},\mathbf{A}^2\vec{b},\ldots,\mathbf{A}^{N-1}\vec{b}\right\}$. Since $\vec{x}_f$ can be any vector in $\mathbb{R}^4$, it also means that $\vec{x}_f-\mathbf{A}^N x[0]$ can be any vector in $\mathbb{R}^4$. This means that in order to be able to reach any state $\vec{x}_f\in\mathbb{R}^4$, the range (column space) of the matrix $\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}$ has to be all of $\mathbb{R}^4$.

P.S.: Congratulations! You have just derived the condition for "controllability" for systems with linear dynamics. When dealing with a system that evolves over time, we can sometimes influence the behavior of the system through various control inputs (for example, the steering wheel and gas pedal of a car or the rudder of an airplane). It is of great importance to know what states (think positions and velocities of a car or configurations of an aircraft) that our system can be controlled to. Controllability is the ability to control the system to any possible state or configuration.

5. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.